Sub-diffusion equations of fractional order and their fundamental solutions

Francesco Mainardi[1], Antonio Mura[1], Gianni Pagnini[2]
and Rudolf Gorenflo[3]

[1] Department of Physics, University of Bologna, and INFN,
Via Irnerio 46, I-40126 Bologna, Italy
{mainardi,mura}@bo.infn.it

[2] ENEA: National Agency for New Technologies, Energy and the Environment,
Centre "E. Clementel", Via Martiri di Monte Sole 4, I-40129 Bologna, Italy
gianni.pagnini@bologna.enea.it

[3] Department of Mathematics and Informatics, Free University of Berlin,
Arnimallee 3, D-14195 Berlin, Germany
gorenflo@mi.fu-berlin.de

Summary. The time-fractional diffusion equation is obtained by generalizing the standard diffusion equation by using a proper time-fractional derivative of order $1-\beta$ in the Riemann-Liouville (R-L) sense or of order β in the Caputo (C) sense, with $\beta \in (0,1)$. The two forms are equivalent and the fundamental solution of the associated Cauchy problem is interpreted as a probability density of a self-similar non-Markovian stochastic process, related to a phenomenon of sub-diffusion (the variance grows in time sub-linearly). A further generalization is obtained by considering a continuous or discrete distribution of fractional time-derivatives of order less than one. Then the two forms are no longer equivalent. However, the fundamental solution still is a probability density of a non-Markovian process but one exhibiting a distribution of time-scales instead of being self-similar: it is expressed in terms of an integral of Laplace type suitable for numerical computation. We consider with some detail two cases of diffusion of distributed order: the double order and the uniformly distributed order discussing the differences between the R-L and C approaches. For these cases we analyze in detail the behaviour of the fundamental solutions (numerically computed) and of the corresponding variance (analytically computed) through the exhibition of several plots. While for the R-L and for the C cases the fundamental solutions seem not to differ too much for moderate times, the behaviour of the corresponding variance for small and large times differs in a remarkable way.

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1 Introduction

The main physical purpose for adopting and investigating diffusion equations of fractional order to describe phenomena of *anomalous diffusion* usually met in transport processes through complex and/or disordered systems including fractal media. In this respect, in recent years interesting reviews, see *e.g.* [39, 40, 47, 59], have appeared, to which (and references therein) we refer the interested reader.

All the related models of random walk turn out to be beyond the classical Brownian motion, which is known to provide the microscopic foundation of the standard diffusion, see *e.g.* [27, 55]. The diffusion-like equations containing fractional derivatives in time and/or in space are usually adopted to model phenomena of anomalous transport in physics, so a detailed study of their solutions is required.

Our attention in this paper will be focused on the time-fractional diffusion equations of a single or distributed order less than 1, which are known to be models for sub-diffusive processes.

Since in the literature we find two different forms for the time-fractional derivative, namely the one in the Riemann-Liouville (R-L) sense, the other in the Caputo (C) sense, we will study the corresponding time-fractional diffusion equations separately. Specifically, we have worked out how to express their fundamental solutions in terms of an integral of Laplace type suitable for a numerical evaluation. Furthermore we have considered the time evolution of the variance for the R-L and C cases. It is known that for large times the variance characterizes the type of anomalous diffusion.

The plan of the paper is as follows.

In Section 2, after having shown the equivalence of the two forms for the time-fractional diffusion equation of a *single order*, namely the R-L form and the C form, we recall the main results for the common fundamental solution, which are obtained by applying two different strategies in inverting its Fourier-Laplace transform. Both techniques yield the fundamental solution in terms of special function of the Wright type that turns out to be self-similar through a definite space-time scaling relationship.

In Section 3 we apply the second strategy for obtaining the fundamental solutions of the time-fractional diffusion equation of *distributed order* in the R-L and C forms, assuming a general order density. We provide for these solutions a representation in terms of a Laplace-type integral of a Fox-Wright function that appears suitable for a numerical evaluation in finite space-time domains. We also provide the general expressions for the Laplace transforms of the corresponding variance.

Then, in Section 4, we consider two case-studies for the fractional diffusion of distributed order: as a discrete distribution we take two distinct orders β_1, β_2 with $0 < \beta_1 < \beta_2 \leq 1$; as continuous distribution we take the uniform density with $0 < \beta < 1$. For these cases we provide the graphical representation of the fundamental solutions (in space at fixed times) and of the evolution in time of the corresponding variance.

Finally, in Section 5, the main conclusions are drawn and directions for future work are outlined.

In order to have a self-contained treatment, we have edited three Appendices: the Appendix A is devoted to the basic notions of fractional calculus, whereas Appendices B and C deal special functions of Mittag-Leffler and Exponential Integral type, respectively, in view of their relevance for our treatment.

2 Time-fractional diffusion of single order

2.1 The standard diffusion

The *standard diffusion equation* in re-scaled non-dimensional variables is known to be

$$\frac{\partial}{\partial t} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t)\,, \quad x \in \mathbf{R}\,, \quad t \in \mathbf{R}_0^+\,, \tag{2.1}$$

with $u(x,t)$ as the field variable. We assume that $u(x,t)$ is subjected to the initial condition

$$u(x,0^+) = u_0(x)\,, \tag{2.2}$$

where $u_0(x)$ denotes a given ordinary or generalized function defined on $\mathbf{R}$, that we assume to be Fourier transformable in ordinary or generalized sense, respectively. We assume to work in a suitable space of generalized functions where it is possible to deal freely with delta functions, integral transforms of Fourier, Laplace and Mellin type, and fractional integrals and derivatives.

It is well known that the fundamental solution (or *Green function*) of Eq. (2.1) i.e. the solution subjected to the initial condition $u(x,0^+) = u_0(x) = \delta(x)$, and to the decay to zero conditions for $|x| \to \infty$, is the Gaussian *probability density function* (*pdf*)

$$u(x,t) = \frac{1}{2\sqrt{\pi}}\, t^{-1/2}\, \mathrm{e}^{-x^2/(4t)}\,, \tag{2.3}$$

that evolves in time with second moment growing linearly with time,

$$\mu_2(t) := \int_{-\infty}^{+\infty} x^2\, u(x,t)\, dx = 2t\,, \tag{2.4}$$

consistently with a law of *normal diffusion*[4]. We note the *scaling property* of the Green function, expressed by the equation

$$u(x,t) = t^{-1/2}\, U(x/t^{1/2})\,, \quad \text{with} \quad U(x) := u(x,1)\,. \tag{2.5}$$

The function $U(x)$ depending on the single variable x turns out to be an even function $U(x) = U(|x|)$ and is called the *reduced Green function.* The variable $X := x/t^{1/2}$ acts as the similarity variable. It is known that the Cauchy problem $\{(2.1) - (2.2)\}$ is equivalent to the integro-differential equation

$$u(x,t) = u_0(x) + \int_0^t \left[\frac{\partial^2}{\partial x^2} u(x,\tau)\right] d\tau\,, \tag{2.6}$$

where the initial condition is incorporated.

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[4] The centred second moment provides the variance usually denoted by $\sigma^2(t)$. It is a measure for the spatial spread of $u(x,t)$ with time of a random walking particle starting at the origin $x = 0$, pertinent to the solution of the diffusion equation (2.1) with initial condition $u(x,0) = \delta(x)$. The asymptotic behaviour of the variance as $t \to \infty$ is relevant to distinguish *normal diffusion* ($\sigma^2(t)/t \to c$, $c > 0$) from anomalous processes of *sub-diffusion* ($\sigma^2(t)/t \to 0$) and of *super-diffusion* ($\sigma^2(t)/t \to +\infty$).

2.2 The two forms of time-fractional diffusion

Now, by using the tools of the fractional calculus we can generalize the above Cauchy problem in order to obtain the so-called *time-fractional diffusion equation* in the two distinct (but mathematically equivalent) forms available in the literature, where the initial condition is understood as (2.2). For the essentials of fractional calculus we refer the interested reader to the Appendix A.

If β denotes a real number such that $0 < \beta < 1$ the two forms are as follows:

$$\frac{\partial}{\partial t} u(x,t) = {}_tD^{1-\beta} \frac{\partial^2}{\partial x^2} u(x,t)\,, \quad x \in \mathbf{R}\,,\; t \in \mathbf{R}_0^+\,; \quad u(x,0^+) = u_0(x)\,, \tag{2.7}$$

where ${}_tD^{1-\beta}$ denotes the *Riemann-Liouville* (R-L) time-derivative of order $1-\beta$ and

$${}_tD_*^{\beta}\, u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t)\,, \quad x \in \mathbf{R}\,,\; t \in \mathbf{R}_0^+\,; \quad u(x,0^+) = u_0(x)\,, \tag{2.8}$$

where ${}_tD_*^{\beta}$ denotes the time derivative of order β intended in the *Caputo* sense. In analogy with the standard diffusion equation we can provide an integro-differential form that incorporates the initial condition (2.2): for this purpose we replace in (2.6) the ordinary integral with the Riemann-Liouville time-fractional integral ${}_tJ^{\beta}$ of order β namely,

$$u(x,t) = u_0(x) + {}_tJ^{\beta} \left[\frac{\partial^2}{\partial x^2} u(x,t) \right]. \tag{2.9}$$

In view of the definitions of ${}_tJ^{\beta}$, ${}_tD^{1-\beta} := {}_tD^1\, {}_tJ^{\beta}$ and ${}_tD_*^{\beta} := {}_tJ^{1-\beta}\, {}_tD^1$, see Appendix A and take there $m = 1$, the above equations read explicitly:

$$\frac{\partial}{\partial t} u(x,t) = \frac{1}{\Gamma(\beta)} \frac{\partial}{\partial t} \left\{ \int_0^t \left[\frac{\partial^2}{\partial x^2} u(x,\tau) \right] \frac{d\tau}{(t-\tau)^{1-\beta}} \right\},\; u(x,0^+) = u_0(x)\,, \tag{2.7$'$}$$

$$\frac{1}{\Gamma(1-\beta)} \int_0^t \left[\frac{\partial}{\partial \tau} u(x,\tau) \right] \frac{d\tau}{(t-\tau)^{\beta}} = \frac{\partial^2}{\partial x^2} u(x,t)\,,\; u(x,0^+) = u_0(x)\,, \tag{2.8$'$}$$

$$u(x,t) = u_0(x) + \frac{1}{\Gamma(\beta)} \int_0^t \left[\frac{\partial^2}{\partial x^2} u(x,\tau) \right] \frac{d\tau}{(t-\tau)^{1-\beta}}\,. \tag{2.9$'$}$$

The two Cauchy problems (2.7), (2.8) and the integro-differential equation (2.9) are equivalent[5]: for example, we derive (2.7) from (2.9) simply differentiating both sides

[5] The integro-differential equation (2.9) was investigated via Mellin transforms by Schneider & Wyss [53] in their pioneering 1989 paper. The time-fractional diffusion equation in the form (2.8) with the Caputo derivative has been preferred and investigated by several authors. From the earlier contributors let us quote Caputo himself [4], Mainardi, see *e.g.* [29, 30, 31] and Gorenflo & Rutman [24]. In particular, Mainardi has expressed the fundamental solution in terms of a special function (of Wright type) of which he has studied the analytical properties and provided plots also for $1 < \beta < 2$, see also [19, 20, 34] and references therein. For the form (2.7) with the R-L derivative earlier contributors include Nigmatullin [45], Giona & Roman [17], the group of Prof. Nonnenmacher, see *e.g.* [38], and Saichev & Zaslavsky [49]. The equivalence between the two forms (2.7) and (2.8) was also pointed out recently by Sokolov and Klafter, see *e.g.* [55].

of (2.9), whereas we derive (2.9) from (2.8) by fractional integration of order β. In fact, in view of the semigroup property (A.2) of the fractional integral, we note that

$$ {}_tJ^{\beta}\, {}_tD_*^{\beta} u(x,t) = {}_tJ^{\beta}\, {}_tJ^{1-\beta}\, {}_tD^{1} u(x,t) = {}_tJ^{1}\, {}_tD^{1} u(x,t) = u(x,t) - u_0(x) . \quad (2.10) $$

In the limit $\beta = 1$ we recover the well-known diffusion equation (2.1).

Eq. (2.7) can be put in a *conservative form* as a *continuity equation*,

$$ \frac{\partial}{\partial t}\, u(x,t) + \frac{\partial}{\partial x}\, F\,[u(x,t)] = 0\,, \quad (2.11) $$

where F is the *flux* given by

$$ F\,[u(x,t)] = -\frac{\partial}{\partial x}\left[{}_tD^{1-\beta}\, u(x,t)\right] = -\frac{\partial}{\partial x}\left\{\frac{1}{\Gamma(\beta)}\frac{\partial}{\partial t}\left[\int_0^t \frac{u(x,\tau)\, d\tau}{(t-\tau)^{1-\beta}}\right]\right\} . \quad (2.12) $$

For $\beta = 1$ in (2.12) we recover in the limit the standard *Fick law*

$$ F\,[u(x,t)] = -\frac{\partial}{\partial x}\, u(x,t)\,, \quad (2.13) $$

which leads to the standard diffusion equation (2.1) by using the continuity law (2.11).

We also note that Eq. (2.12) can be interpreted as a generalized Fick law[6] where (long) memory effects are taken into account through a time-fractional derivative of order $1-\beta$ in the *Riemann-Liouville* sense.

We observe that the form (2.7) of the time-fractional diffusion equation with the R-L fractional derivative has the advantage of being derived in a direct way from a conservation principle by introducing a generalized Fick's law: in addition it can be interpreted as a master evolution equation of a dynamical system where in the LHS the time derivative of the first order usually appears.

The form (2.8) with the Caputo derivative, however, has the advantage to be treated in a simpler way with the Laplace transform requiring as the initial value $u(x,0^+)$ as in the standard case, see Eq. (A.13). We note how in its definition (A.6) (for $m=1$) the first derivative is weighted by a memory function of power law type, that formally degenerates to a delta function ($\delta(t) = t_+^{-1}/\Gamma(0)$, see [15]) as soon as the order tends to 1 from below. We observe that the Caputo form can be obtained from the master integral equation of the Continuous Time Random Walk (CTRW) by a well scaled transition to the diffusion limit as shown by Gorenflo and Mainardi, [22, 23], see also [36, 51].

2.3 The fundamental solution

Let us consider from now on the Eq. (2.8) with $u_0(x) = \delta(x)$: the fundamental solution can be obtained by applying in sequence the Fourier and Laplace transforms to it. We write, for generic functions $v(x)$ and $w(t)$, these transforms as follows:

[6] We recall that the Fick law is essentially a phenomenological law, which represents the simplest relationship between the flux F and the gradient of the concentration u. If u is a temperature, F is the heat-flux, so we speak of the Fourier law. In both cases the law can be replaced by a more suitable phenomenological relationship which may account for possible non-local, non-linear and memory effects, without violating the conservation law expressed by the continuity equation.

$$\begin{aligned} \mathcal{F}\{v(x);\kappa\} &= \widehat{v}(\kappa) := \int_{-\infty}^{+\infty} \mathrm{e}^{i\kappa x}\, v(x)\, dx\,,\ \kappa \in \mathbf{R}\,,\\ \mathcal{L}\{w(t);s\} &= \widetilde{w}(s) := \int_{0}^{+\infty} \mathrm{e}^{-st}\, w(t)\, dt\,,\ s \in \mathbf{C}\,. \end{aligned} \tag{2.14}$$

Then, in the Fourier-Laplace domain our Cauchy problem [(2.8) with $u(x,0^+) = \delta(x)$], after applying formula (A.13) for the Laplace transform of the fractional derivative and observing $\widehat{\delta}(\kappa) \equiv 1$, appears in the form

$$s^{\beta}\, \widehat{\widetilde{u}}(\kappa,s) - s^{\beta-1} = -\kappa^2\, \widehat{\widetilde{u}}(\kappa,s)\,,$$

implying

$$\widehat{\widetilde{u}}(\kappa,s) = \frac{s^{\beta-1}}{s^{\beta}+\kappa^2}\,, \quad 0 < \beta \le 1\,, \qquad \Re(s) > 0\,, \quad \kappa \in \mathbf{R}\,. \tag{2.15}$$

To determine the Green function $u(x,t)$ in the space-time domain we can follow two alternative strategies related to the order in carrying out the inversions in (2.15).
(S1) : invert the Fourier transform getting $\widetilde{u}(x,s)$ and then invert the remaining Laplace transform;
(S2) : invert the Laplace transform getting $\widehat{u}(\kappa,t)$ and then invert the remaining Fourier transform.

Strategy (S1): Recalling the Fourier transform pair, see *e.g.* [1],

$$\frac{a}{b+\kappa^2} \stackrel{\mathcal{F}}{\leftrightarrow} \frac{a}{2b^{1/2}}\, \mathrm{e}^{-|x| b^{1/2}}\,, \quad b > 0\,, \tag{2.16}$$

and setting $a = s^{\beta-1}$, $b = s^{\beta}$ we get

$$\widetilde{u}(x,s) = \frac{s^{\beta/2-1}}{2}\, \mathrm{e}^{-|x| s^{\beta/2}}\,, \quad 0 < \beta \le 1\,. \tag{2.17}$$

The strategy (S1) has been applied by Mainardi [29, 30, 31] to obtain the Green function in the form

$$u(x,t) = t^{-\beta/2}\, U\left(|x|/t^{\beta/2}\right)\,, \quad -\infty < x < +\infty\,, \quad t \ge 0\,, \tag{2.18}$$

where the variable $X := x/t^{\beta/2}$ acts as *similarity variable* and the function $U(x) := u(x,1)$ denotes the *reduced Green function.* Restricting from now on our attention to $x \ge 0$, the solution turns out as

$$\begin{aligned} U(x) = U(-x) &= \frac{1}{2}\, M_{\frac{\beta}{2}}(x) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-x)^k}{k!\, \Gamma[-\beta k/2 + (1-\beta/2)]}\\ &= \frac{1}{2\pi} \sum_{k=0}^{\infty} \frac{(-x)^k}{k!}\, \Gamma[(\beta(k+1)/2]\, \sin[(\pi\beta(k+1)/2]\,, \end{aligned} \tag{2.19}$$

where $M_{\frac{\beta}{2}}(x)$ is an an entire transcendental function (of order $1/(1-\beta/2)$) of the Wright type, see also [19, 20] and [48].

Strategy (S2): Recalling the Laplace transform pair, see e.g. [13, 21, 48],

$$\frac{s^{\beta-1}}{s^{\beta}+c} \stackrel{\mathcal{L}}{\leftrightarrow} E_{\beta}(-ct^{\beta})\,, \quad c > 0\,, \tag{2.20}$$

and setting $c = \kappa^2$ we get

$$\widehat{u}(\kappa,t) = E_\beta(-\kappa^2 t^\beta)\,, \quad 0 < \beta \le 1\,, \tag{2.21}$$

where E_β denotes the Mittag-Leffler function, see Appendix B.

The strategy (S2) has been followed by Gorenflo, Iskenderov & Luchko [18] and by Mainardi, Luchko & Pagnini [33] to obtain the Green functions of the more general space-time-fractional diffusion equations, and requires to invert the Fourier transform by using the machinery of the Mellin convolution and the Mellin-Barnes integrals. Restricting ourselves here to recall the final results, the reduced Green function for the time-fractional diffusion equation now appears, for $x \ge 0$, in the form:

$$U(x) = U(-x) = \frac{1}{\pi}\int_0^\infty \cos(\kappa x)\, E_\beta\left(-\kappa^2\right) d\kappa = \frac{1}{2x}\,\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(1-s)}{\Gamma(1-\beta s/2)}\, x^{\,s}\, ds, \tag{2.22}$$

with $0 < \gamma < 1$. By solving the Mellin-Barnes integrals using the residue theorem, we arrive at the same power series (2.19).

Both strategies allow us to prove that the Green function is non-negative and normalized, so it can be interpreted as a spatial probability density evolving in time with the similarity law (2.18). Although the two strategies are equivalent for yielding the required result, the second one appears more general and so more suitable to treat the more complicated case of fractional diffusion of distributed order, see the next Section.

It is relevant to point out, see *e.g.* [30, 33], that for $0 < \beta < 1$ as $|x| \to \infty$ the solution decays faster than exponential and slower than Gaussian. We have, for $x > 0$,

$$U(x) \sim A\, x^a\, \mathrm{e}^{-bx^c}\,, \quad x \to \infty\,, \tag{2.23}$$

with

$$A = \left\{2\pi(2-\beta)\, 2^{\beta/(2-\beta)} \beta^{(2-2\beta)/(2-\beta)}\right\}^{-1/2}\,, \tag{2.24}$$

$$a = \frac{2\beta-2}{2(2-\beta)}\,, \quad b = (2-\beta)\, 2^{-2/(2-\beta)} \beta^{\beta/(2-\beta)}\,, \quad c = \frac{2}{2-\beta}\,. \tag{2.25}$$

We note in fact that c increases from 1 to 2 as β varies from 0 to 1; for $\beta = 1$ we recover the exact solution $U(x) = \exp(-x^2/4)/(2\sqrt{\pi})$, consistent with (2.3). Furthermore, the moments (of even order) of $u(x,t)$ are

$$\mu_{2n}(t) := \int_{-\infty}^{+\infty} x^{2n}\, u(x,t)\, dx = \frac{\Gamma(2n+1)}{\Gamma(\beta n+1)} t^{\beta n}\,, \quad n = 0,1,2,\ldots\,,\ t \ge 0\,. \tag{2.26}$$

Of particular interest is the evolution of the variance $\sigma^2(t) = \mu_2(t)$ (the second centred moment); we get from (2.26):

$$\sigma^2(t) = 2\,\frac{t^\beta}{\Gamma(\beta+1)}\,, \quad 0 < \beta \le 1\,, \tag{2.27}$$

so that for $\beta < 1$ we note a sub-linear growth in time, consistent with an anomalous process of *slow diffusion* in contrast with the law (2.3) of normal diffusion. Such result can also be obtained in a simpler way from the Fourier transform (2.21) noting that

$$\sigma^2(t) = -\frac{\partial^2}{\partial\kappa^2}\,\widehat{u}(\kappa=0,t)\,. \tag{2.28}$$

2.4 Graphical representation of the fundamental solutions

Let us consider the time-fractional diffusion equation of a single order $\beta = \beta_0$, whose fundamental solution has the peculiar property to be self-similar according to the similarity variable $x/t^{\beta_0/2}$. For this reason it is sufficient to consider the fundamental solution for $t = 1$, namely the reduced Green function $U(x)$, given by Eq. (2.19) in terms of the special function $M_{\beta_0/2}(x)$ of the Wright type. In Fig.1 we show the

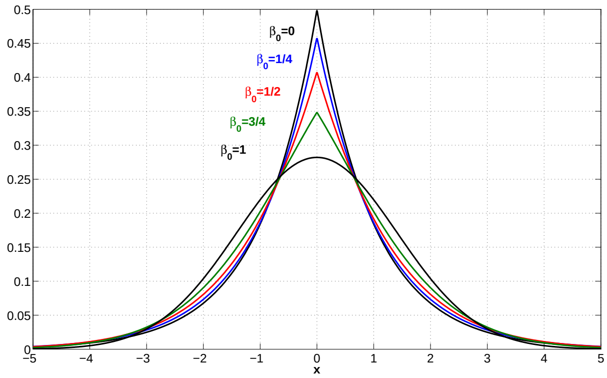

Fig. 1. Plots (in linear scales) of the reduced Green function $U(x) = \frac{1}{2} M_{\beta_0/2}(x))$ versus x (in the interval $|x| \leq 5$), for $\beta_0 = 0, 1/4, 1/2, 3/4, 1$.

graphical representations of $U(x)$ for different orders β_0 ranging from $\beta_0 = 0$, for which we recover the Laplace *pdf*

$$U(x) = \frac{1}{2}\, \mathrm{e}^{-|x|}\,, \tag{2.29}$$

to $\beta_0 = 1$, for which we recover the Gaussian *pdf* (of variance $\sigma^2 = 2$)

$$U(x) = \frac{1}{2\sqrt{\pi}}\, \mathrm{e}^{-x^2/4}\,. \tag{2.30}$$

To visualize the decay of the queues of the above (symmetric) *pdf*'s as stated in Eqs. (2.23)-(2.25) we refer to Fig. 2, where we have adopted semi-logarithmic scales. In this case the decay-plot of the queues is ranging from a straight line ($\beta_0 = 0$) to a parabolic line ($\beta = 1$).

For more information about plots and properties of the M-Wright function we refer the reader to previous articles of our research group, see *e.g.* [30, 31, 33, 34].

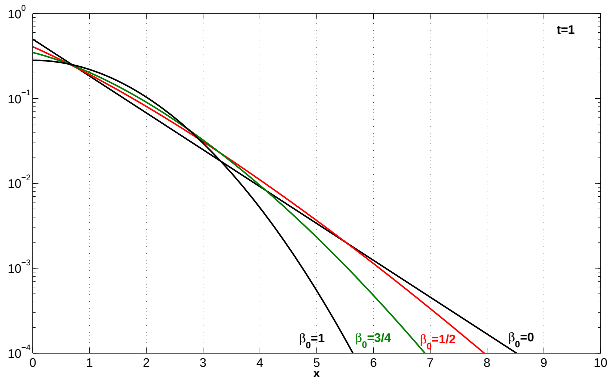

Fig. 2. Plots (in linear-logarithmic scales) of the reduced Green function $U(x) = \frac{1}{2}M_{\beta_0/2}(x)$) versus x (in the interval $0 \le x \le 10$), for $\beta_0 = 0, 1/2, 3/4, 1$.

3 Time-fractional diffusion equation of distributed order

3.1 The two forms for time-fractional diffusion

The time-fractional diffusion equations (2.7) and (2.8) can be generalized by using the notion of time-fractional derivative of distributed order[7]. For this purpose we need to consider a function $p(\beta)$ that acts as weight for the order of differentiation $\beta \in (0, 1]$ such that

$$p(\beta) \geq 0\,, \quad \text{and} \quad \int_0^1 p(\beta)\, d\beta = c > 0\,. \tag{3.1}$$

The positive constant c can be taken as 1 if we like to assume the normalization condition for the integral. Clearly, some special conditions of regularity and behaviour near the boundaries will be required for the weight function $p(\beta)$[8]. Such function, that can be referred to as the *order density* if $c = 1$, is allowed to have δ-components if we are interested in a discrete distribution of orders.

Then, if we weight the time-fractional derivative in Eq. (2.7) (where it is intended in the R-L sense), and in Eq. (2.8) (where it is intended the C sense) by using the weight function $p(\beta)$ in (3.1), we finally obtain the *time-fractional diffusion equation of distributed order* in the two forms:

$$\frac{\partial}{\partial t}\, u(x,t) = \int_0^1 p(\beta)\, {}_tD^{1-\beta} \left[\frac{\partial^2}{\partial x^2}\, u(x,t) \right] d\beta\,, \quad x \in \mathbf{R},\; t \geq 0\,, \tag{3.2}$$

[7] We find an earlier idea of fractional derivative of distributed order in time in the 1969 book by Caputo [4], that was later developed by Caputo himself, see [5, 6], and by Bagley & Torvik, see [2].

[8] For the weight function $p(\beta)$ we conveniently require that its primitive $P(\beta) = \int_0^\beta p(\beta')\, d\beta'$ vanishes at $\beta = 0$ and is there continuous from the right, attains the value c at $\beta = 1$ and has at most finitely many (upwards) jump points in the half-open interval $0 < \beta \leq 1$, these jump points allowing delta contributions to $p(\beta)$ (particularly relevant for discrete distributions of orders).

and

$$\int_0^1 p(\beta)\, \left[{}_tD_*^{\beta}\, u_*(x,t) \right] \, d\beta \,=\, \frac{\partial^2}{\partial x^2}\, u_*(x,t)\,, \quad x \in \mathbf{R},\; t \geq 0\,. \tag{3.3}$$

From now on we shall restrict our attention on the fundamental solutions of Eqs. (3.2)-(3.3) so we understand that these equations are subjected to the initial condition $u(x,0^+) = u_*(x,0^+) = \delta(x)$. Since for distributed order the solution depends on the selected approach (as we shall show hereafter), we now distinguish the fractional equations (3.2) and (3.3) and their fundamental solutions by decorating in the Caputo case the variable $u(x,t)$ with subscript $*$.

Diffusion equations of distributed order of C-type (3.3) have recently been discussed in [9, 10, 11, 54] and in [44]. Diffusion equations of distributed order of R-L type (2.2) have been considered by Sokolov et al. [54, 55]. These authors have referred to Eqs. (3.3), (3.2) as to *normal* and *modified* forms of the time-fractional diffusion equation of distributed order, respectively. In their analysis they have pointed out the different evolutions of the variance corresponding to the case of a combination of two derivatives of order β_1, β_2 with $0 < \beta_1 < \beta_2 < 1$, although both forms exhibit slow diffusion. For the modified form with two fractional orders, recently Langlands [28] has provided the fundamental solution as an infinite series of H-Fox functions.

As usual, we have considered the initial condition $u(x,0^+) = u_*(x,0^+) = \delta(x)$ in order to keep the probability meaning. Indeed, already in the paper [9], it was shown that the Green function is non-negative and normalized, so allowing interpretation as a density of the probability at time t of a diffusing particle to be in the point x. The main interest of the authors in [9, 10, 11, 54] was devoted to the second moment of the Green function (the displacement variance) in order to show the sub-diffusive character of the related stochastic process by analyzing some interesting cases of the order-density function $p(\beta)$. In this paper, extending the approach by Naber [44], we are interested to provide a general representation of the fundamental solution corresponding to a generic order-density $p(\beta)$.

For a thorough general study of fractional pseudo-differential equations of distributed order let us cite the paper by Umarov and Gorenflo [57]. For a relationship between the C fractional diffusion equation of distributed order (3.3) and the Continuous Time Random Walk (CTRW) models we may refer to the paper by Gorenflo and Mainardi [23]. Let us remark that the flux formula (2.12) for the fractional diffusion equations of a single order (2.7)-(2.8) can be generalized to hold for the R-L fractional diffusion equation of distributed order (3.2) as follows:

$$F\left[u(x,t)\right] = -\frac{\partial}{\partial x}\left\{ \int_0^1 p(\beta)\, \left[{}_tD^{1-\beta}\, u(x,t) \right]\, d\beta \right\}\,. \tag{3.4}$$

3.2 The Fourier-Laplace transforms of the fundamental solutions

Let us now apply the Laplace transform to Eqs. (3.2)-(3.3) by using the rules (A.15) and (A.13) appropriate to the R-L and C derivatives, respectively, with $m = 1$. Introducing the relevant functions

$$A(s) = s \int_0^1 p(\beta)\, s^{-\beta}\, d\beta\,, \tag{3.5}$$

and

$$B(s) = \int_0^1 p(\beta)\, s^{\beta}\, d\beta\,, \tag{3.6}$$

we then get for the R-L and C cases, after simple manipulation, the Laplace transforms of the corresponding fundamental solutions:

$$\widehat{\widetilde{u}}(\kappa, s) = \frac{1}{s + \kappa^2 A(s)}\,, \tag{3.7}$$

and

$$\widehat{\widetilde{u}}_*(\kappa, s) = \frac{B(s)/s}{\kappa^2 + B(s)}\,. \tag{3.8}$$

We easily note that in the particular case $p(\beta) = \delta(\beta - \beta_0)$ we have in (3.5): $A(s) = s^{1-\beta_0}$, and in (3.6): $B(s) = s^{\beta_0}$. Then, Eqs. (3.7) and (3.8) provide the same result (2.15) and its consequences for the time-fractional diffusion of a single order $\beta = \beta_0$.

3.3 The inversion of the Laplace transforms

By inverting the Laplace transforms in (3.7) and (3.8) we obtain the remaining Fourier transforms of the fundamental solutions for the R-L and C time-fractional diffusion of distributed order.

Let us start with the R-L case. We get (in virtue of the Titchmarsh theorem on Laplace inversion) the representation

$$\widehat{u}(\kappa, t) = -\frac{1}{\pi} \int_0^{\infty} \mathrm{e}^{-rt}\, \mathrm{Im}\left\{\widehat{\widetilde{u}}\left(\kappa, r\mathrm{e}^{i\pi}\right)\right\}\, dr\,, \tag{3.9}$$

that requires the expression of $-\mathrm{Im}\left\{1/[s + \kappa^2 A(s)]\right\}$ along the ray $s = r\,\mathrm{e}^{i\pi}$ with $r > 0$ (the branch cut of the function $s^{-\beta}$). We write

$$A\left(r\,\mathrm{e}^{i\pi}\right) = \rho\, \cos(\pi\gamma) + i\rho \sin(\pi\gamma)\,, \tag{3.10}$$

where

$$\begin{cases} \rho = \rho(r) = \left|A\left(r\,\mathrm{e}^{i\pi}\right)\right|\,, \\ \gamma = \gamma(r) = \dfrac{1}{\pi} \arg\left[A\left(r\,\mathrm{e}^{i\pi}\right)\right]. \end{cases} \tag{3.11}$$

Then, after simple calculations, we get

$$\widehat{u}(\kappa, t) = \int_0^{\infty} \frac{\mathrm{e}^{-rt}}{r}\, H(r; \kappa)\, dr\,, \tag{3.12}$$

with

$$H(\kappa; r) = \frac{1}{\pi}\, \frac{\kappa^2\, r\, \rho\, \sin(\pi\gamma)}{r^2 - 2\kappa^2\, r\, \rho\, \cos(\pi\gamma) + \kappa^4 \rho^2} \ge 0\,. \tag{3.13}$$

Similarly for the C case we obtain

$$\widehat{u}_*(\kappa, t) = -\frac{1}{\pi} \int_0^{\infty} \mathrm{e}^{-rt}\, \mathrm{Im}\left\{\widehat{\widetilde{u}}_*\left(\kappa, r\mathrm{e}^{i\pi}\right)\right\}\, dr\,, \tag{3.14}$$

that requires the expression of $-\mathrm{Im}\left\{B(s)/[s(\kappa^2 + B(s))]\right\}$ along the ray $s = r\,\mathrm{e}^{i\pi}$ with $r > 0$ (the branch cut of the function s^{β}). We write

$$B\left(r\,\mathrm{e}^{i\pi}\right) = \rho_*\,\cos(\pi\gamma_*) + i\rho_*\,\sin(\pi\gamma_*)\,, \tag{3.15}$$

where

$$\begin{cases} \rho_* = \rho_*(r) = \left|B\left(r\,\mathrm{e}^{i\pi}\right)\right|\,, \\ \gamma_* = \gamma_*(r) = \dfrac{1}{\pi}\arg\left[B\left(r\,\mathrm{e}^{i\pi}\right)\right]\,. \end{cases} \tag{3.16}$$

After simple calculations we get

$$\widehat{u}_*(\kappa,t) = \int_0^\infty \frac{\mathrm{e}^{-rt}}{r}\,K(r;\kappa)\,dr\,, \tag{3.17}$$

with

$$K(\kappa;r) = \frac{1}{\pi}\,\frac{\kappa^2\,\rho_*\,\sin(\pi\gamma_*)}{\kappa^4 + 2\kappa^2\,\rho_*\,\cos(\pi\gamma_*) + \rho_*^2} \ge 0\,. \tag{3.18}$$

We note that the expressions of H and K are related through the transformation

$$\rho_* \Longleftrightarrow r/\rho\,, \quad \gamma_* \Longleftrightarrow 1-\gamma\,. \tag{3.19}$$

3.4 The inversion of the Fourier transforms

Since $u(x,t)$ and $u_*(x,t)$ are symmetric in x, the inversion formula for the Fourier transforms in (3.12) and (3.17) yields

$$u(x,t) = \frac{1}{\pi}\int_0^{+\infty}\cos(\kappa x)\left\{\int_0^\infty \frac{\mathrm{e}^{-rt}}{r}\,H(\kappa,r)\,dr\right\}d\kappa\,, \tag{3.20}$$

and

$$u_*(x,t) = \frac{1}{\pi}\int_0^{+\infty}\cos(\kappa x)\left\{\int_0^\infty \frac{\mathrm{e}^{-rt}}{r}\,K(\kappa,r)\,dr\right\}d\kappa\,. \tag{3.21}$$

We note that the evaluation of the Fourier integral in Eq. (3.21) concerning the C case has been recently carried out by Mainardi and Pagnini [35] by the method of the Mellin transform. Referring the reader to [35] for details we can now state that for the C case the fundamental solution reads, taking as usual $x \ge 0$,

$$u_*(x,t) = \frac{1}{2\pi x}\int_0^\infty \frac{e^{-rt}}{r}F_*(\rho_*^{1/2}x)\,dr\,, \tag{3.22}$$

where

$$F_*(\rho_*^{1/2}x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma(1-s)\sin(\pi\gamma_* s/2)(\rho_*^{1/2}x)^s ds\,. \tag{3.23}$$

and $\rho_* = \rho_*(r)\,,\ \gamma_* = \gamma_*(r)\,.$ In [35] the Authors have expressed the function F_* in terms of Fox-Wright functions by using the method of the Mellin-Barnes integrals; then the series expansion of F_* yields the required solution as

$$u_*(x,t) = \frac{1}{2\pi}\sum_{k=0}^{\infty}\frac{(-x)^k}{k!}\,\varphi_k(t)\,, \quad x \ge 0\,, \tag{3.24}$$

with

$$\varphi_k(t) = \int_0^\infty \frac{e^{-rt}}{r} \sin[\pi\gamma_*(k+1)/2]\, \rho_*^{(k+1)/2}\, dr\,. \qquad (3.25)$$

For numerical purposes we now prefer to find an alternative representation of the function F_* that we can get by taking inspiration from an exercise in the book by Paris and Kaminski, see [46]: p.89, Eq. (3.3.2). The new representation of F_* reads

$$\begin{aligned} F_*(\rho_*^{1/2}x) &= \mathrm{Im}\{\rho_*^{1/2} x\, \mathrm{e}^{i\pi\gamma_*/2} \mathrm{e}^{-e^{i\pi\gamma_*/2}\rho_*^{1/2}x}\} \\ &= \rho_*^{1/2} x\, e^{-\rho_*^{1/2} x \cos(\pi\gamma_*/2)} \sin[\pi\gamma_*/2 - \rho^{1/2} x \sin(\pi\gamma_*/2)]\,. \end{aligned} \qquad (3.26)$$

As a matter of fact, from the numerical view-point the integral representation (3.22) with (3.26) is indeed more convenient than the series representation (3.24) with (3.25) that was provided in [35].

For the fundamental solution of the R-L case, we shall use the representation (3.22) with (3.26) of the C case by invoking the transformation (3.19).

3.5 The variance of the fundamental solutions

We now consider the evaluation of the variance of the fundamental solutions, that is, according to the two approaches:

$$\text{R-L}\ :\ \sigma^2(t) := \int_{-\infty}^{+\infty} x^2\, u(x,t)\, dx\,; \quad \text{C}\ :\ \sigma_*^2(t) := \int_{-\infty}^{+\infty} x^2\, u_*(x,t)\, dx\,. \qquad (3.27)$$

Like for the single order case we can obtain these quantities (fundamental for classifying the type of diffusion) in a simpler way according

$$\text{R-L}\ :\ \sigma^2(t) = -\frac{\partial^2}{\partial\kappa^2}\, \widehat{u}(\kappa=0,t)\,; \quad \text{C}\ :\ \sigma_*^2(t) = -\frac{\partial^2}{\partial\kappa^2}\, \widehat{u}_*(\kappa=0,t)\,. \qquad (3.28)$$

As a consequence of (3.28) we thus must invert only Laplace transforms as follows. We have, for κ near zero,
for the R-L case we get from Eq. (3.7),

$$\widehat{\widetilde{u}}(\kappa,s) = \frac{1}{s}\left(1 - \kappa^2\, \frac{A(s)}{s} + \ldots\right), \text{ so } \widetilde{\sigma^2}(s) = -\frac{\partial^2}{\partial\kappa^2}\widehat{\widetilde{u}}(\kappa=0,s) = \frac{2A(s)}{s^2}\,, \qquad (3.29)$$

for the C case we get from Eq. (3.8)

$$\widehat{\widetilde{u}}_*(\kappa,s) = \frac{1}{s}\left(1 - \kappa^2\, \frac{1}{B(s)} + \ldots\right), \text{ so } \widetilde{\sigma^2}(s) = -\frac{\partial^2}{\partial\kappa^2}\widehat{\widetilde{u}}(\kappa=0,s) = \frac{2}{s\,B(s)}. \qquad (3.30)$$

4 Examples of fractional diffusion of distributed order

We shall now concentrate our interest to choices of some typical weight functions $p(\beta)$ in (3.1) that characterizes the order distribution for the time-fractional diffusion equations of distributed order (3.2) and (3.3). This will allow us to compare the results for the R-L form and for the C form.

4.1 The fractional diffusion of double-order

First, we consider the choice

$$p(\beta) = p_1 \delta(\beta - \beta_1) + p_2 \delta(\beta - \beta_2) \,, \quad 0 < \beta_1 < \beta_2 \le 1 \,, \tag{4.1}$$

where the constants p_1 and p_2 are both positive, conveniently restricted to the normalization condition $p_1 + p_2 = 1$.

Then for the R-L case we have

$$A(s) = p_1 \, s^{1-\beta_1} + p_2 \, s^{1-\beta_2} \,, \tag{4.2}$$

so that, inserting (4.2) in (3.7),

$$\widehat{\widetilde{u}}(\kappa, s) = \frac{1}{s[1 + \kappa^2 (p_1 \, s^{-\beta_1} + p_2 \, s^{-\beta_2})]} \,, \tag{4.3}$$

Similarly, for the C case we have

$$B(s) = p_1 \, s^{\beta_1} + p_2 \, s^{\beta_2} \,, \tag{4.4}$$

so that, inserting (4.4) in (3.8),

$$\widehat{\widetilde{u}}_*(\kappa, s) = \frac{p_1 \, s^{\beta_1} + p_2 \, s^{\beta_2}}{s[\kappa^2 + p_1 \, s^{\beta_1} + p_2 \, s^{\beta_2}]} \,. \tag{4.5}$$

We leave as an exercise the derivation of the spectral functions $H(\kappa; r)$ and $K(\kappa; r)$ of the corresponding fundamental solutions, that are used for the numerical computation.

Let us now evaluate the second moments starting from the corresponding Laplace transforms (3.29) and (3.30) inserting the expressions of $A(s)$ and $B(s)$) provided by Eqs. (4.2) and (4.4), respectively.
For the R-L form we have

$$\widetilde{\sigma^2}(s) = 2 \, p_1 \, s^{-(1+\beta_1)} + 2 \, p_2 \, s^{-(1+\beta_2)} \,; \tag{4.6}$$

for the C form we have

$$\widetilde{\sigma^2}_*(s) = \frac{2}{p_1 \, s^{(1+\beta_1)} + p_2 \, s^{(1+\beta_2)}} \,, \tag{4.7}$$

Now the Laplace inversion yields:
for the R-L case, see and compare Sokolov et al [54] and Langlands [28],

$$\sigma^2(t) = 2 \, p_1 \, \frac{t^{\beta_1}}{\Gamma(\beta_1 + 1)} + 2 \, p_2 \, \frac{t^{\beta_2}}{\Gamma(\beta_2 + 1)} \sim \begin{cases} 2p_1 \, \dfrac{t^{\beta_1}}{\Gamma(1+\beta_1)} \,; & t \to 0^+ \,, \\ 2p_2 \, \dfrac{t^{\beta_2}}{\Gamma(1+\beta_2)} \,, & t \to +\infty \,; \end{cases} \tag{4.8}$$

for the C case, see and compare Chechkin et al. [9]

$$\sigma_*^2(t) = \frac{2}{p_2}\, t^{\beta_2}\, E_{\beta_2-\beta_1,\beta_2+1}\left(-\frac{p_1}{p_2}\, t^{\beta_2-\beta_1}\right) \sim \begin{cases} \dfrac{2}{p_2}\, \dfrac{t^{\beta_2}}{\Gamma(1+\beta_2)}\,, & t \to 0^+\,, \\ \dfrac{2}{p_1}\, \dfrac{t^{\beta_1}}{\Gamma(1+\beta_1)}\,, & t \to +\infty\,. \end{cases} \tag{4.9}$$

Then we see that for the R-L case we have an explicit combination of two power laws: the smallest exponent(β_1) dominates at small times whereas the largest exponent (β_2) dominates at large times. For the C case we have a Mittag-Leffler function in two parameters so we have a combination of two power laws only asymptotically for small and large times; precisely we get a behaviour opposite to the previous one, so the largest exponent(β_2) dominates at small times whereas the smallest exponent (β_1) dominates at large times.

We can derive the above asymptotic behaviours directly from the Laplace transforms (4.6)-(4.7) by applying the Tauberian theory for Laplace transforms[9]. In fact for the R-L case we note that for $A(s)$ in (4.2) $s^{1-\beta_1}$ is negligibly small in comparison with $s^{1-\beta_2}$for $s \to 0^+$ and, viceversa, $s^{1-\beta_2}$ is negligibly small in comparison to $s^{1-\beta_1}$for $s \to +\infty$. Similarly for the C case we note that for $B(s)$ in (4.4) s^{β_2} is negligibly small in comparison to $s^{\beta_1}$for $s \to 0^+$ and, viceversa, s^{β_1} is negligibly small in comparison $s^{\beta_2}$for $s \to +\infty$.

4.2 The fractional diffusion of uniformly distributed order

Second, we consider the choice

$$p(\beta) = 1\,, \quad 0 < \beta < 1\,. \tag{4.10}$$

For the R-L case we have

$$A(s) = s\int_0^1 s^{-\beta}\, d\beta = \frac{s-1}{\log s}\,, \tag{4.11}$$

hence, inserting (4.11) in (3.7)

$$\widehat{\widetilde{u}}(\kappa, s) = \frac{\log s}{s\log s + \kappa^2\,(s-1)}\,. \tag{4.12}$$

For the C case we have

$$B(s) = \int_0^1 s^{\beta}\, d\beta = \frac{s-1}{\log s}\,, \tag{4.13}$$

hence, inserting (4.13) in (3.8),

$$\widehat{\widetilde{u}}_*(\kappa, s) = \frac{1}{s}\,\frac{s-1}{\kappa^2\log s + s - 1} = \frac{1}{s} - \frac{1}{s}\,\frac{\kappa^2\log s}{\kappa^2\log s + s - 1}\,. \tag{4.14}$$

We leave as an exercise the derivation of the spectral functions $H(\kappa; r)$ and $K(\kappa; r)$ of the corresponding fundamental solutions, that are used for the numerical computation.

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[9] According to this theory the asymptotic behaviour of a function $f(t)$ near $t = \infty$ and $t = 0$ is (formally) obtained from the asymptotic behaviour of its Laplace transform $\widetilde{f}(s)$ for $s \to 0^+$ and for $s \to +\infty$, respectively.

Let us now evaluate the second moments starting from the corresponding Laplace transforms (3.29) and (3.30) inserting the expressions of $A(s)$ and $B(s)$) provided by Eqs. (4.2) and (4.4), respectively. We note that for this special order distribution we have $A(s) = B(s)$.
For the R-L case we have

$$\widetilde{\sigma^2}(s) = 2\left[\frac{1}{s\log s} - \frac{1}{s^2\log s}\right] . \tag{4.15}$$

Then, by inversion, see Appendix C: Eq. (C.16), we get

$$\sigma^2(t) = 2\,[\nu(t,0) - \nu(t,1)] \sim \begin{cases} 2/\log(1/t)\,, & t \to 0, \\ 2t/\log t, & t \to \infty\,, \end{cases} \tag{4.16}$$

where

$$\nu(t,a) = \int_0^\infty \frac{t^{a+\tau}}{\Gamma(a+\tau+1)}\,d\tau\,, \quad a > -1\,,$$

denotes a special function introduced in Appendix C along with its Laplace transform.
For the C case we have

$$\widetilde{\sigma^2_*}(s) = \frac{2}{s}\,\frac{\log s}{s-1} . \tag{4.17}$$

Then, by inversion, see Appendix C: Eqs. (C.11), (C.14), and compare with Chechkin et al. [9], Eqs. (23)-(27),

$$\sigma^2_*(t) = 2\left[\log t + \gamma + \mathrm{e}^{\,t}\,\mathcal{E}_1(t)\right] \sim \begin{cases} 2t\log(1/t), & t \to 0, \\ 2\log(t), & t \to \infty\,, \end{cases} \tag{4.18}$$

where $\mathcal{E}(t)$ denotes the exponential integral function and $\gamma = 0.57721...$ is the so-called Euler-Mascheroni constant.

4.3 Graphical representation of the fundamental solutions

For the general time-fractional diffusion equations of distributed order, namely (3.2) for the R-L form and (3.3) for the C form, we limit ourselves to a few cases selected from the examples treated above. Specifically we consider the case of two distinct, equally weighted, orders, see Eq. (4.1) with $p_1 = p_2 = 1/2$ and the case of a uniform distribution of orders, see Eq. (4.10).

We recall that, in contrast with the single order, for the distributed order the self-similarity of the fundamental solution is lost so we need to provide graphical representations for different times.

For the case of two orders, we chose $\{\beta_1 = 1/4, \beta_2 = 1\}$ in order to better contrast the different evolution of the fundamental solution for the R-L and the C forms. In Fig. 3 we exhibit the plots of the corresponding solution versus x (in the interval $|x| \le 5$), at different times, selected as $t = 0.1$, $t = 1$ and $t = 10$. In this limited spatial range we can note how the time evolution of the *pdf* depends on the different time-asymptotic behaviour of the variance, for the two forms, as stated in Eqs. (4.8)-(4.9), respectively.

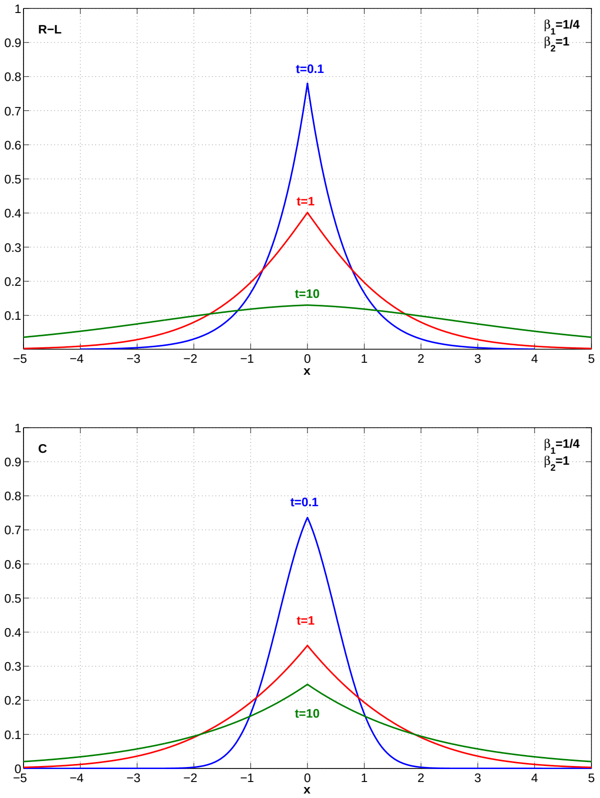

Fig. 3. Plots of the fundamental solution versus x (in the interval $|x| \leq 5$), for the case $\{\beta_1 = 1/4, \beta_2 = 1\}$ at times $t = 0.1, 1, 10$; top: R-L form, bottom: C form.)

For the uniform distribution, we find it instructive to compare in Fig. 4 the solutions corresponding to R-L and C forms with the solutions of the fractional diffusion of a single order $\beta_0 = 1/4, 3/4, 1$ at fixed times, selected as $t = 1, 10$. We have skipped the order $\beta_0 = 1/2$ and the time $t = 0.1$ for a better view of the plots. Then in Figs 5,6 we compare the variance of for moderate times ($0 \leq t \leq 10$, using linear scales) and large times ($10^1 \leq t \leq 10^7$, using logarithmic scales), respectively. Here we have inserted the plot for $\beta_0 = 1/2$.

To interpret these asymptotic behaviours we observe that $\beta = 0$ is the smallest, $\beta = 1$ the largest relevant index for the constant order-density. Due to the logarithmic constituents in the R-L case the smallest β, namely 0, now plays the dominant role for $s \to \infty$ and $t \to 0$, see (4.11),(4.15) and (4.16), whereas the largest β, namely 1, is dominant for $s \to 0$ and $t \to \infty$. This situation is reversed for the C case, see (4.13),(4.17) and (4.18). We observe that in the R-L case the variance (the second moment) grows slightly slower than linearly for $t \to \infty$, but extremely slowly near $t = 0$. In the C case the variance exhibits a slightly super-linear growth near $t = 0$, but an extremely slow growth for $t \to \infty$.

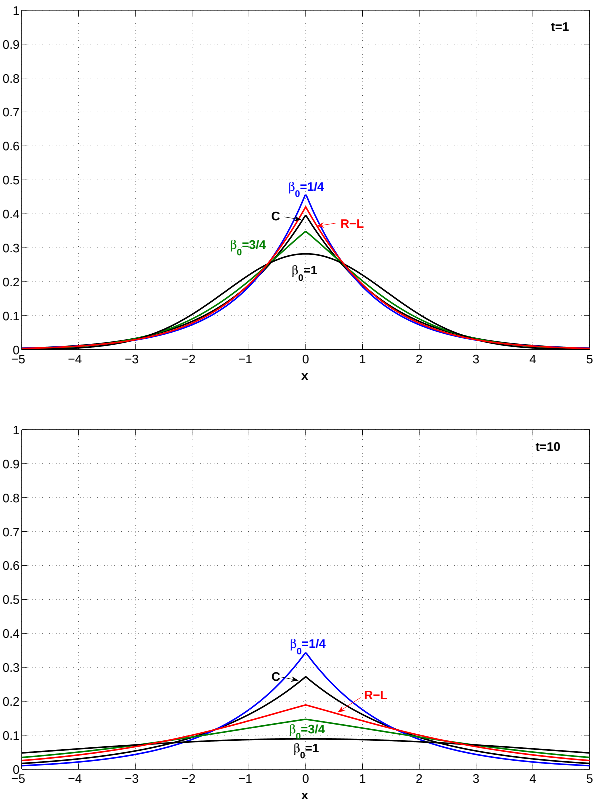

Fig. 4. Plots of the fundamental solution versus x (in the interval $|x| \le 5$), for the uniform order distribution in R-L and C forms compared with the solutions of some cases of single order at $t = 1$ (top) and $t = 10$ (bottom).

5 Conclusions and outlook

After outlining the basic theory of the Cauchy problem for the spatially one-dimensional and symmetric time-fractional diffusion equation (with its main equivalent formulations), we have paid special attention to transform methods for finding its fundamental solution or (exploiting self-similarity) the corresponding reduced Green function. We have stressed the importance of the transforms of Fourier, Laplace and Mellin and of the functions of Mittag-Leffler and Wright type, avoiding however the cumbersome H-Fox function notations.

A natural first step for construction of the fundamental solution consists in applying in either succession the transforms of Fourier in space and Laplace in time to the Cauchy problem. This yields in the Fourier-Laplace domain the solution in explicit form, but for the space-time domain we must invert both transforms in sequence for which there are two choices, both leading to the same power series in the spatial variable with time-dependent coefficients. The strategy, called by us strategy (S2), of first doing Laplace inversion and then the Fourier inversion yields

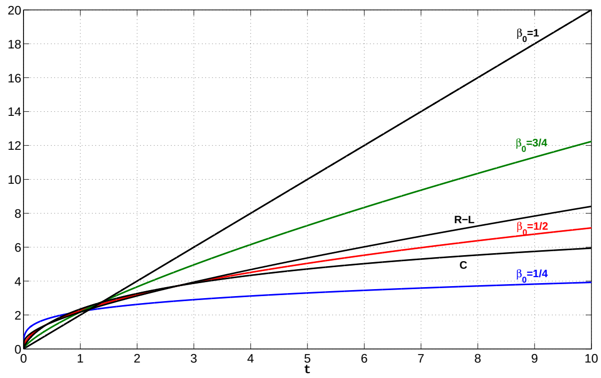

Fig. 5. Plots of the variance versus t in the interval $0 \leq t \leq 10$ (linear scales), for the uniform order distribution.

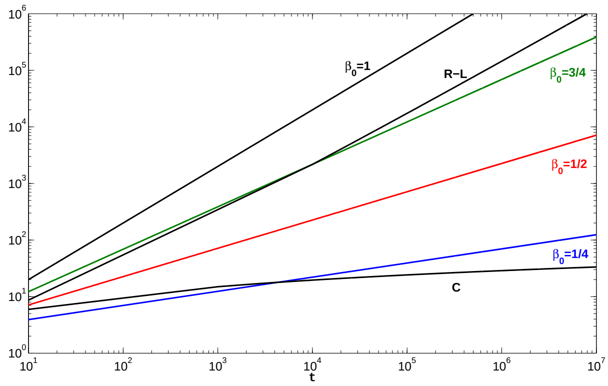

Fig. 6. Plots of the variance versus t in the interval $10^1 \leq t \leq 10^7$ (logarithmic scales) for the uniform order distribution

the reduced Green function as a Mellin-Barnes integral form which, by the calculus of residues, the power series is obtained. This strategy can be adapted to the treatment of the more general case of the time-fractional diffusion equation of distributed order. Now the fundamental solution can be expressed as an integral over a Mellin-Barnes integral containing two parameters having the form of functionals of the order-density. Again for the fundamental solution a power series comes out whose coefficients, however, are time-dependent functionals of the order-density. But, if there is more than one time derivative-order present, self-similarity is lost. Finally, we have worked out how to express the fundamental solution in terms of an integral of Laplace type, more suitable for a numerical evaluation.

We have studied in detail and illustrated by graphics the time-fractional diffusion of a single order (where self-similarity holds true) and two simple but noteworthy case-studies of distributed order, namely the case of a superposition of two different orders β_1 and β_2 and the case of a uniform order distribution. In the first case one of the orders dominates the time-asymptotics near zero, the other near infinity, but β_1 and β_2 change their roles when switching from the (R-L) form to the (C) form of the time-fractional diffusion. The asymptotics for uniform order density is remarkably different, the extreme orders now being (roughly speaking) 0 and 1. We now meet super-slow and slightly super-fast time behaviours of the variance near zero and near infinity, again with the interchange of behaviours between the R-L and C form. We clearly see the above effects described in the figures at the end of Section 4, in particular the extremely slow growth of the variance as $t \to \infty$ for the C form.

More general studies are desirable for fractional diffusion equations of distributed order in time as well as in space. For the case of one single order in time and in space (the space-time-fractional diffusion equation) we refer the reader to the exhaustive paper by Mainardi, Luchko & Pagnini [33].

As our emphasis in this paper is on pure analysis we have not touched the wide field of simulations of trajectories of a particle subjected to the random process modelled by the equations at hand.

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Appendix A: Essentials of fractional calculus

For a sufficiently well-behaved function $f(t)$ $(t \in \mathbf{R}^+)$ we may define the fractional derivative of order μ $(m-1 < \mu \le m\,,\ m \in \mathbf{N})$, see *e.g.* [21, 48], in two different senses, that we refer here as to *Riemann-Liouville* (R-L) derivative and *Caputo* (C) derivative, respectively. Both derivatives are related to the so-called Riemann-Liouville fractional integral of order $\alpha > 0$ defined as

$$ {}_tJ^{\alpha}\, f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}\, f(\tau)\, d\tau\,, \quad \alpha > 0\,. \qquad (A.1) $$

We recall the convention ${}_tJ^0 = I$ (Identity operator) and the semigroup property

$$ {}_tJ^{\alpha}\, {}_tJ^{\beta} = {}_tJ^{\beta}\, {}_tJ^{\alpha} = {}_tJ^{\alpha+\beta}\,, \quad \alpha, \beta \ge 0\,. \qquad (A.2) $$

Furthermore

$$ {}_tJ^{\alpha}\, t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)}\, t^{\gamma+\alpha}\,, \quad \alpha \ge 0\,, \quad \gamma > -1\,, \quad t > 0\,. \qquad (A.3) $$

The fractional derivative of order $\mu > 0$ in the *Riemann-Liouville* sense is defined as the operator ${}_tD^{\mu}$ which is the left inverse of the Riemann-Liouville integral of order μ (in analogy with the ordinary derivative), that is

$$ {}_tD^{\mu}\, {}_tJ^{\mu} = I\,, \quad \mu > 0\,. \qquad (A.4) $$

If m denotes the positive integer such that $m-1<\mu\le m$, we recognize from Eqs. (A.2) and (A.4) ${}_tD^\mu\, f(t) := {}_tD^m\, {}_tJ^{m-\mu}\, f(t)$, hence

$$ {}_tD^\mu\, f(t) = \begin{cases} \dfrac{d^m}{dt^m}\left[\dfrac{1}{\Gamma(m-\mu)}\displaystyle\int_0^t \dfrac{f(\tau)\,d\tau}{(t-\tau)^{\mu+1-m}}\right], & m-1<\mu<m, \\ \dfrac{d^m}{dt^m} f(t)\,, & \mu=m. \end{cases} \qquad (A.5) $$

For completion we define ${}_tD^0 = I$.

On the other hand, the fractional derivative of order $\mu>0$ in the *Caputo* sense is defined as the operator ${}_tD_*^\mu$ such that ${}_tD_*^\mu\, f(t) := {}_tJ^{m-\mu}\, {}_tD^m\, f(t)$, hence

$$ {}_tD_*^\mu\, f(t) = \begin{cases} \dfrac{1}{\Gamma(m-\mu)}\displaystyle\int_0^t \dfrac{f^{(m)}(\tau)\,d\tau}{(t-\tau)^{\mu+1-m}}, & m-1<\mu<m, \\ \dfrac{d^m}{dt^m} f(t)\,, & \mu=m. \end{cases} \qquad (A.6) $$

Thus, when the order is not integer the two fractional derivatives differ in that the derivative of order m does not generally commute with the fractional integral.

We point out that the *Caputo* fractional derivative satisfies the relevant property of being zero when applied to a constant, and, in general, to any power function of non-negative integer degree less than m, if its order μ is such that $m-1<\mu\le m$. Furthermore we note that

$$ {}_tD^\mu\, t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\mu)}\, t^{\gamma-\mu}\,, \quad \mu\ge 0\,, \quad \gamma>-1\,, \quad t>0\,. \qquad (A.7) $$

Gorenflo and Mainardi [21] have shown the essential relationships between the two fractional derivatives (when both of them exist),

$$ {}_tD_*^\mu\, f(t) = \begin{cases} {}_tD^\mu \left[f(t) - \displaystyle\sum_{k=0}^{m-1} f^{(k)}(0^+)\,\dfrac{t^k}{k!}\right], \\ {}_tD^\mu\, f(t) - \displaystyle\sum_{k=0}^{m-1} \dfrac{f^{(k)}(0^+)\, t^{k-\mu}}{\Gamma(k-\mu+1)}, \end{cases} \quad m-1<\mu<m\,. \qquad (A.8) $$

In particular, if $m=1$ we have

$$ {}_tD_*^\mu\, f(t) = \begin{cases} {}_tD^\mu \left[f(t) - f(0^+)\right], \\ {}_tD^\mu\, f(t) - \dfrac{f(0^+)\, t^{-\mu}}{\Gamma(1-\mu)}, \end{cases} \quad 0<\mu<1\,. \qquad (A.9) $$

The *Caputo* fractional derivative, practically ignored in the mathematical treatises, represents a sort of regularization in the time origin for the *Riemann-Liouville* fractional derivative. We note that for its existence all the limiting values $f^{(k)}(0^+) := \lim\limits_{t\to 0^+} f(t)$ are required to be finite for $k=0,1,2.\dots m-1$.

We observe the different behaviour of the two fractional derivatives at the end points of the interval $(m-1,m)$ namely when the order is any positive integer: whereas ${}_tD^\mu$ is, with respect to its order μ, an operator continuous at any positive integer, ${}_tD_*^\mu$ is an operator left-continuous since

$$\begin{cases} \displaystyle\lim_{\mu\to(m-1)^+} {}_tD_*^\mu\, f(t) \;=\; f^{(m-1)}(t) - f^{(m-1)}(0^+)\,, \\ \displaystyle\lim_{\mu\to m^-} {}_tD_*^\mu\, f(t) \;=\; f^{(m)}(t)\,. \end{cases} \qquad (A.10)$$

We also note for $m-1<\mu\le m$,

$${}_tD^\mu\, f(t) \;=\; {}_tD^\mu\, g(t) \iff f(t) = g(t) + \sum_{j=1}^{m} c_j\, t^{\mu-j}\,, \qquad (A.11)$$

$${}_tD_*^\mu\, f(t) \;=\; {}_tD_*^\mu\, g(t) \iff f(t) = g(t) + \sum_{j=1}^{m} c_j\, t^{m-j}\,. \qquad (A.12)$$

In these formulae the coefficients c_j are arbitrary constants. Last but not least, we point out the major utility of the Caputo fractional derivative in treating initial-value problems for physical and engineering applications where initial conditions are usually expressed in terms of integer-order derivatives. This can be easily seen using the Laplace transformation, according to which

$$\mathcal{L}\left\{{}_tD_*^\mu\, f(t);s\right\} = s^\mu\, \widetilde{f}(s) - \sum_{k=0}^{m-1} s^{\mu-1-k}\, f^{(k)}(0^+)\,, \quad m-1<\mu\le m\,, \qquad (A.13)$$

where $\widetilde{f}(s) \;=\; \mathcal{L}\left\{f(t);s\right\} \;=\; \displaystyle\int_0^\infty \mathrm{e}^{-st}\, f(t)\, dt\,,\ s\in \mathbf{C}$, and $f^{(k)}(0^+) \;:=\; \displaystyle\lim_{t\to 0^+} f(t)$. The corresponding rule for the Riemann-Liouville derivative is more cumbersome: for $m-1<\mu\le m$ it reads

$$\mathcal{L}\left\{{}_tD^\mu\, f(t);s\right\} = s^\mu\, \widetilde{f}(s) - \sum_{k=0}^{m-1} \left[{}_tD^k\, {}_tJ^{(m-\mu)}\right]\, f(0^+)\, s^{m-1-k}\,, \qquad (A.14)$$

where, in analogy with (A.13), the limit for $t\to 0^+$ is understood to be taken after the operations of fractional integration and derivation. As soon as all the limiting values $f^{(k)}(0^+)$ are finite and $m-1<\mu<m$, the formula (A.14) simplifies into

$$\mathcal{L}\left\{{}_tD^\mu\, f(t);s\right\} = s^\mu\, \widetilde{f}(s)\,. \qquad (A.15)$$

In the special case $f^{(k)}(0^+)=0$ for $k=0,1,m-1$, we recover the identity between the two fractional derivatives, consistently with Eq. (A.8).

We remind that the Laplace transform rule (A.13) was practically the starting point of Caputo himself in defining his generalized derivative in the late sixties, [3, 4]. Later, Caputo and Mainardi in 1971 [7, 8] and Mainardi in the nineties, see e.g. [29, 30], have followed the notation involving a convolution with the so-called Gel'fand-Shilov (generalized) function $\Phi_\lambda(t) := t_+^{\lambda-1}/\Gamma(\lambda)$ discussed in [15]. The notation here adopted was introduced in a systematic way by Gorenflo and Mainardi in their 1996 CISM lectures [21], partly based on the book on Abel Integral Equations by Gorenflo & Vessella [25] and on the article by Gorenflo & Rutman [24].

For further reading on the theory and applications of fractional calculus we recommend to consult in addition to the well-known books by Samko, Kilbas & Marichev [50], by Miller & Ross [41], by Podlubny [48], those appeared in the last few years, by Kilbas, Srivastava & Trujillo [26], by West, Bologna & Grigolini [58], and by Zaslavsky [60].

Appendix B: The Mittag-Leffler functions

B.1 The classical Mittag-Leffler function

Let us recall that the Mittag-Leffler function $E_\mu(z)$ $(\mu > 0)$ is an entire transcendental function of order $1/\mu$, defined in the complex plane by the power series

$$E_\mu(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu\, k+1)}\,, \quad \mu > 0\,, \quad z \in \mathbf{C}\,. \qquad (B.1)$$

It was introduced and studied by the Swedish mathematician Mittag-Leffler at the beginning of the XX century to provide a noteworthy example of entire function that generalizes the exponential (to which it reduces for $\mu = 1$). For details on this function we refer e.g. to [13, 26, 21, 48, 50]. In particular we note that the function $E_\mu(-x)$ $(x \geq 0)$ turns a completely monotonic function of x if $0 < \mu \leq 1$. This property is still valid if we consider the variable $x = \lambda\, t^\mu$ where λ is a positive constant. Thus the function $E_\mu(-\lambda t^\mu)$ preserves the *complete monotonicity* of the exponential $\exp(-\lambda t)$: indeed it is represented in terms of a real Laplace transform (of a real parameter r) of a non-negative function (that we refer to as the spectral function)

$$E_\mu(-\lambda t^\mu) = \frac{1}{\pi} \int_0^\infty \frac{\mathrm{e}^{-rt}}{r}\, \frac{\lambda r^\mu\, \sin(\mu\pi)}{\lambda^2 + 2\lambda\, r^\mu\, \cos(\mu\pi) + r^{2\mu}}\, dr\,,\ t \geq 0\,,\ 0 < \mu < 1\,. \quad (B.2)$$

We note that as $\mu \to 1$ the spectral function tends to the generalized Dirac function $\delta(r-\lambda)$. We point out that the Mittag-Leffler function (B.2) starts at $t = 0$ as a stretched exponential and decreases for $t \to \infty$ like a power with exponent $-\mu$:

$$E_\mu(-\lambda t^\mu) \sim \begin{cases} 1 - \lambda\, \dfrac{t^\mu}{\Gamma(1+\mu)} \sim \exp\left\{-\dfrac{\lambda t^\mu}{\Gamma(1+\mu)}\right\}, & t \to 0^+\,, \\ \dfrac{t^{-\mu}}{\lambda\, \Gamma(1-\mu)}\,, & t \to \infty\,. \end{cases} \qquad (B.3)$$

The noteworthy results (B.2) and (B.3) can also be derived from the Laplace transform pair

$$\mathcal{L}\{E_\mu(-\lambda t^\mu); s\} = \frac{s^{\mu-1}}{s^\mu + \lambda}\,. \qquad (B.4)$$

In fact it it sufficient to apply the Titchmarsh theorem ($s = r\mathrm{e}^{i\pi}$) for deriving (B.2) and the Tauberian theory ($s \to \infty$ and $s \to 0$) for deriving (B.3).

If $\mu = 1/2$ we have for $t \geq 0$:

$$E_{1/2}(-\lambda\sqrt{t}) = \mathrm{e}^{\lambda^2 t}\, \mathrm{erfc}(\lambda\sqrt{t}) \sim 1/(\lambda\sqrt{\pi\, t})\,, \quad \text{as} \quad t \to \infty\,, \qquad (B.5)$$

where erfc denotes the *complementary error* function, see e.g. [1].

B.2 The generalized Mittag-Leffler function

The Mittag-Leffler function in two parameters $E_{\mu,\nu}(z)$ ($\Re\{\mu\} > 0,\ \nu \in \mathbf{C}$) is defined by the power series

$$E_{\mu,\nu}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu\, k+\nu)}\,, \quad z \in \mathbf{C}\,. \qquad (B.6)$$

It generalizes the classical Mittag-Leffler function to which it reduces for $\nu = 1$. It is an entire transcendental function of order $1/\Re\{\mu\}$ on which the reader can inform himself by again consulting *e.g.* [13, 26, 21, 48, 50].

With $\mu, \nu \in \mathbf{R}$ the function $E_{\mu,\nu}(-x)$ $(x \geq 0)$ turns a completely monotonic function of x if $0 < \mu \leq 1$ and $\nu \geq \mu > 0$, see *e.g.* [52, 42, 43]. Again this property is still valid if we consider the variable $x = \lambda\, t^\mu$ where λ is a positive constant.

We point out the Laplace transform pair, see [48],

$$\mathcal{L}\{t^{\nu-1}\, E_{\mu,\nu}(-\lambda t^\mu); s\} = \frac{s^{\mu-\nu}}{s^\mu+\lambda}\,, \quad \mu > 0\,,\ \nu > 0\,. \qquad (B.7)$$

For $0 < \mu \leq \nu < 1$ this Laplace transform pair can be used to derive for the function $E_{\mu,\nu}(-\lambda t^\mu)$ its asymptotic representations as $t \to 0$ and $t \to \infty$, by applying the Tauberian theory ($s \to \infty$, $s \to 0$). Indeed we have

$$E_{\mu,\nu}(-\lambda t^\mu) \sim \begin{cases} \dfrac{1}{\Gamma(\nu)}\left[1-\lambda\,\dfrac{\Gamma(\nu)}{\Gamma(\nu+\mu)}\,t^\mu\right] \sim \dfrac{1}{\Gamma(\nu)}\exp\left\{-\dfrac{\lambda\Gamma(\nu)t^\mu}{\Gamma(1+\mu)}\right\}, & t \to 0^+, \\ \dfrac{1}{\lambda}\,\dfrac{t^{-\mu+\nu-1}}{\Gamma(\nu-\mu)}\,, & t \to \infty. \end{cases} \qquad (B.8)$$

In particular, for $0 < \mu = \nu < 1$ we point out the noteworthy identity

$$t^{-(1-\mu)}\, E_{\mu,\mu}\left(-\lambda\, t^\mu\right) = -\frac{1}{\lambda}\,\frac{d}{dt} E_\mu\left(-\lambda\, t^\mu\right)\,. \qquad (B.9)$$

Appendix C: The Exponential integral functions

C.1 Basic definitions and properties

The exponential integral function, that we denote by $\mathcal{E}_1(z)$, is defined as

$$\mathcal{E}_1(z) = \int_z^\infty \frac{\mathrm{e}^{-t}}{t}\, dt = \int_1^\infty \frac{\mathrm{e}^{-zt}}{t}\, dt\,. \qquad (C.1)$$

We have used the letter $\mathcal{E}$ instead of E (commonly adopted in the literature) in order to avoid confusion with the Mittag-Leffler functions that play a more relevant role in fractional calculus. This function exhibits a branch cut along the negative real semi-axis and admits the representation

$$\mathcal{E}_1(z) = -\gamma - \log z - \sum_{n=1}^{\infty} \frac{z^n}{n\, n!}\,, \quad |\arg z| < \pi\,, \qquad (C.2)$$

where $\gamma = 0.57721...$ is the so-called Euler-Mascheroni constant. The power series in the R.H.S. is absolutely convergent in all of $\mathbf{C}$ and represents the entire function called the *modified exponential integral*[10]

$$\mathrm{Ein}\,(z) := \int_0^z \frac{1-\mathrm{e}^{-\zeta}}{\zeta}\, d\zeta = -\sum_{n=1}^{\infty} \frac{z^n}{n\, n!}\,, \qquad (C.3)$$

Thus, in view of (C.2) and (C.3), we write

$$\mathcal{E}_1(z) = -\gamma - \log z + \mathrm{Ein}\,(z)\,, \quad |\arg z| < \pi\,. \qquad (C.4)$$

This relation is important for understanding the analytic properties of the classical exponential integral function in that it isolates the multi-valued part represented by the logarithmic function from the regular part represented by the entire function $\mathrm{Ein}\,(z)$. Furthermore, $\mathrm{Ein}\,(x)$ is an *increasing* function on $\mathbf{R}$ because

$$\frac{d}{dx}\,\mathrm{Ein}\,(x) = \frac{1-\mathrm{e}^{-x}}{x} > 0\,, \quad \forall\, x \in \mathbf{R}\,.$$

In $\mathbf{R}^+$ the function $\mathrm{Ein}\,(x)$ turns out to be a *Bernstein function*, which means that is positive, increasing, with the first derivative *completely monotonic.*

C.2 Asymptotic expansion of the exponential integral

The asymptotic behaviour as $z \to \infty$ of the exponential integrals can be obtained from the integral representation (C.1) noticing that

$$\mathcal{E}_1(z) := \int_z^{\infty} \frac{\mathrm{e}^{-t}}{t}\, dt = \mathrm{e}^{-z} \int_0^{\infty} \frac{\mathrm{e}^{-u}}{u+z}\, du\,. \qquad (C.5)$$

In fact, by repeated partial integrations in the R.H.S., we get

$$\mathcal{E}_1(z) \sim \frac{\mathrm{e}^{-z}}{z} \sum_{n=0}^{\infty} (-1)^n \frac{n!}{z^n}\,, \quad z \to \infty\,, \quad |\arg z| \le \pi - \delta\,. \qquad (C.6)$$

C.3 Laplace transform pairs related to exponential integrals

We now report a number of relevant Laplace transform pairs related to logarithmic and exponential integral functions.

Taking $t > 0$, the basic Laplace transforms pairs are

$$\mathcal{L}\{\log t; s\} = -\frac{\gamma + \log s}{s}\,, \quad \Re s > 0\,, \qquad (C.7)$$

$$\mathcal{L}\{\mathcal{E}_1(t); s\} = \frac{\log\,(s+1)}{s}\,, \quad \Re s > -1\,, \qquad (C.8)$$

[10] The Italian mathematicians Tricomi and Gatteschi have pointed out the major utility of the modified exponential integral, being an entire function, with respect to the exponential integral. This entire function was introduced by S.A. Schelkunoff in 1944. ["Proposed symbols for the modified cosine and integral exponential integral", *Quart. Appl. Math.* **2** (1944), p. 90]

The proof of (C.7) and (C.8) is found, for example, in the treatise by Ghizzetti & Ossicini, see [16], Eqs. [4.6.15-16]), pp. 104-105. We then easily derive

$$\mathcal{L}\{\gamma + \log t; s\} = -\frac{\log s}{s}\,, \quad \Re s > 0\,, \tag{C.9}$$

$$\mathcal{L}\{\gamma + \log t + \mathcal{E}_1(t); s\} = \frac{\log (s+1)}{s} - \frac{\log s}{s} = \frac{\log (1/s+1)}{s} \quad \Re s > 0\,. \tag{C.10}$$

$$\mathcal{L}\{\gamma + \log t + \mathrm{e}^t\, \mathcal{E}_1(t); s\} = \frac{\log s}{s-1} - \frac{\log s}{s} = \frac{\log s}{s(s-1)}\,, \quad \Re s > 0\,, \tag{C.11}$$

We outline the different asymptotic behaviour of the three functions $f_1(t) = \mathcal{E}_1(t)$, $f_2(t) = \mathrm{Ein}\,(t) = \gamma + \log t + \mathcal{E}_1(t)$ and $f_3(t) = \gamma + \log t + \mathrm{e}^t\, \mathcal{E}_1(t)$ for small argument ($t \to 0^+$) and large argument ($t \to +\infty$). By using Eqs. (C.2), (C.4) and (C.6), we have

$$f_1(t) = \mathcal{E}_1(t)) \sim \begin{cases} \log (1/t)\,, & t \to 0^+\,, \\ \mathrm{e}^{-t}/t\,, & t \to +\infty\,. \end{cases} \tag{C.12}$$

$$f_2(t) = \mathrm{Ein}\,(t) = \gamma + \log t + \mathcal{E}_1(t) \sim \begin{cases} t\,, & t \to 0^+\,, \\ \log t\,, & t \to +\infty\,. \end{cases} \tag{C.13}$$

$$f_3(t) = \gamma + \log t + \mathrm{e}^t\, \mathcal{E}_1(t) \sim \begin{cases} t \log (1/t)\,, & t \to 0^+\,, \\ \log t\,, & t \to +\infty\,. \end{cases} \tag{C.14}$$

We note that all the above asymptotic representations can be obtained from the Laplace transforms of the corresponding functions by invoking the Tauberian theory for *regularly varying functions* (power functions multiplied by *slowly varying functions*[11]), a topic adequately treated in the treatise on Probability by Feller, see [14], Chapter XIII.5.

C.4 The $\nu(t)$ function and the related Laplace transform pair

In the third volume of the Handbook of the Bateman Project, in the Chapter XVIII devoted the Miscellaneous functions, see [13], §18.3 pp. 217-224, we find, in addition to the functions of the Mittag-Leffler and Wright type, the function

$$\nu(t, a) = \int_0^\infty \frac{t^{a+\tau}}{\Gamma(a+\tau+1)}\, d\tau\,, \quad a > -1\,. \tag{C.15}$$

Such special function is relevant for our purposes because of the Laplace transform pair, see [13], Eq. (18), p.222,

$$\mathcal{L}\{\nu(t, a; s\} = \frac{1}{s^{a+1} \log s}\,, \quad \Re s > 0\,, \tag{C.16}$$

[11] **Definition:** We call a (measurable) positive function $a(y)$, defined in a right neighbourhood of zero, *slowly varying at zero* if $a(cy)/a(y) \to 1$ with $y \to 0$ for every $c > 0$. We call a (measurable) positive function $b(y)$, defined in a neighbourhood of infinity, *slowly varying at infinity* if $b(cy)/a(y) \to 1$ with $y \to \infty$ for every $c > 0$. Examples: $(\log y)^\gamma$ with $\gamma \in \mathbf{R}$ and $\exp\,(\log y/\log\, \log y)$.

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