# The asymptotic universality of the Mittag-Leffler waiting time law in continuous time random walks


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### Abstract

We show the asymptotic long-time equivalence of a generic power law waiting time distribution to the Mittag-Leffler distribution, the waiting time distribution characteristic for a time-fractional continuous time random walk. This asymptotic equivalence is effected by a combination of "rescaling" time and "respeeding" the relevant renewal process and subsequent passage to a limit for which we need a suitable relation between the parameters of rescaling and respeeding. As far as we know such procedure has been first applied in the Sixties of the past century by Gnedenko and Kovalenko in their theory of "thinning" a renewal process. Turning our attention to continuous time random walks with a generic power law jump distribution, "rescaling" space can be interpreted as a second kind of "respeeding" which then, again under a proper relation between the relevant parameters leads in the limit to the time-space fractional diffusion equation. Finally, we treat the 'time-fractional drift" process as properly scaled limit of the counting number of a Mittag-Leffler renewal process.

# 1 Introduction

The purpose of this paper is to outline the fundamental role the Mittag-Leffler function in renewal processes that are relevant in the theories of anomalous diffusion. As a matter of fact the interest on this function in statistical physics and probability theory has recently been increased as it shown by the large number of papers published in the last decade of which a brief bibliography[1] includes [2, 3, 15, 20, 21, 27, 28, 29, 30, 31, 33, 38, 41, 44, 48, 56, 58, 63].

In this paper we develop a theory of long-time behaviour of a renewal process with a generic power law waiting distribution of order $\beta$, $0 < \beta \le 1$ (thereby for easy readability dispensing with decoration by a slowly varying function). To bring the distant future into near sight we change the unit of time from 1 to $1/\tau$, $0 < \tau \ll 1$.

For the random waiting times $T$ this means replacing $T$ by $\tau T$. Now, having very many events in a moderate span of time we compensate this compression by respeeding the whole process, actually slowing it down so that again we have a moderate number of events in a moderate span of time. We will relate the rescaling factor $\tau$ and the respeeding factor $a$ in such a way that in the limit $\tau \to 0$ we have a reasonable process, namely one whose waiting time distribution is the Mittag-Leffler distribution whose density is

$$\phi^{ML}(t) = -\frac{d}{dt} E_\beta(-t^\beta)\,, \quad 0 < \beta \le 1\,, \tag{1.1}$$

with the Mittag-Leffler function

$$E_\beta(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + 1)}\,, \quad z \in \mathbf{C}\,, \quad \beta > 0\,. \tag{1.2}$$

Our method is, in some sense, analogous to the one applied in the Sixties of the past century by Gnedenko and Kovalenko [14] in their analysis of *thinning* (or *rarefaction* of a renewal process. They found, under certain power law assumptions, in the infinite thinning limit, for the waiting time density the Laplace transform $1/(1 + s^\beta)$ but did not identify it as a Mittag-Leffler type function.

As we consider our renewal process formally as a continuous random walk (CTRW) with constant non-random jumps 1 in space (for the counting function $N(t)$ we embed ab initio our theory into the CTRW, thus being in the position to treat the theory of a time-fractional CTRW as limiting case of a CTRW with power law waiting time distribution. In this context the pioneering paper by Balakrishnan [1] of 1985 deserves to be mentioned. Balakrishnan also found the importance of the Laplace transform $1/(1 + s^\beta)$ in the time-fractional CTRW and time-fractional diffusion, but also did not identify it as the Laplace transform of $\phi^{ML}(t)$. Then, in 1995 Hilfer and Anton [29] showed that the waiting time density characteristic for time-fractional CTRW is

[1] We are well aware of the fact that the Mittag-Leffler function has a richer history in pure theory of probability as well as in diverse applications than presented here by us. It is our intention to survey this on another occasion. The bibliography of the present paper, even if it refers to papers not necessarily cited in the text, is far to be exhaustive.

expressed in terms of the Mittag-Leffler function in two parameters, that is

$$\phi^{ML}(t) = t^{\beta-1} E_{\beta,\beta}(-t^\beta) , \quad 0 < \beta \le 1 , \tag{1.3}$$

with the generalized Mittag-Leffler function

$$E_{\beta,\gamma}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + \gamma)} , \quad z \in \mathbf{C} , \quad \beta > 0 , \quad \gamma \in \mathbf{R} . \tag{1.4}$$

The form (1.3) is equivalent to (1.2) but these Authors did not explicitly identify their function as simply related to the derivative of the standard Mittag-Leffler function. Note that we have slightly modified their original notation.

Then, assuming a suitable power law also for the spatial jumps of a CTRW, we show that by a rescaling of the jump widths by a positive factor $h$ (that means a change of the unit of space from 1 to $1/h$ to bring into near sight the far-away space) another respeeding is effected, now an acceleration, that in the limit $h \to 0$ (under a proper relation between $h$ and $\tau$) leads to space-time fractional diffusion.

Finally, we pass to a properly scaled limit for the counting function $N(t)$ of renewal process (again under power law assumption) and obtain the time-fractional drift process (viewing $N(t)$ as a spatial variable).

In Appendix A we provide an outline of the thinning theory for renewal processes according to Gnedenko and Kovalenko. Essentials properties of the derivative of fractional order in time and in space are given in Appendix B and Appendix C, respectively. Finally, in Appendix D we give some information on a special function of Wright type for its relevant relations both with Mittag-Leffler function and with the unilateral stable densities in probability theory.

# 2 The continuous time random walk (CTRW)

The name *continuous time random walk* (CTRW) became popular in physics after Montroll, Weiss and Scher (just to cite the pioneers) in the 1960's and 1970's published a celebrated series of papers on random walks for modelling diffusion processes on lattices, see *e.g.* [45, 46], and the book by Weiss [62] with references therein. CTRWs are rather good and general phenomenological models for diffusion, including processes of anomalous transport, that can be understood in the framework of the classical renewal theory, as stated *e.g.* in the booklet by Cox [7]. In fact a CTRW can be considered as a compound renewal process (a simple renewal process with reward) or a random walk *subordinated* to a simple renewal process.

A CTRW is generated by a sequence of independent identically distributed (*iid*) positive random waiting times $T_1, T_2, T_3, \dots$, each having the same probability density function $\phi(t)$ , $t > 0$ , and a sequence of *iid* random jumps $X_1, X_2, X_3, \dots$, in $\mathbf{R}$ , each having the same probability density $w(x)$ , $x \in \mathbf{R}$ .

Let us remark that, for ease of language, we use the word density also for generalized functions in the sense of Gel'fand & Shilov [13], that can be interpreted as probability

measures. Usually the *probability density functions* are abbreviated by *pdf*. We recall that $\phi(t) \geq 0$ with $\int_0^\infty \phi(t)\, dt = 1$ and $w(x) \geq 0$ with $\int_{-\infty}^{+\infty} w(x)\, dx = 1$.
Setting $t_0 = 0\,,\ t_n = T_1 + T_2 + \dots T_n$ for $n \in \mathbf{N}$, the wandering particle makes a jump of length $X_n$ in instant $t_n$, so that its position is $x_0 = 0$ for $0 \leq t < T_1 = t_1$, and $x_n = X_1 + X_2 + \dots X_n$, for $t_n \leq t < t_{n+1}$. We require the distribution of the waiting times and that of the jumps to be independent of each other. So, we have a compound renewal process (a renewal process with reward), compare [7].
By natural probabilistic arguments we arrive at the *integral equation* for the probability density $p(x,t)$ (a density with respect to the variable $x$) of the particle being in point $x$ at instant $t$, see *e.g.* [20, 22, 41, 54, 55, 56],

$$p(x,t) = \delta(x)\, \Psi(t) \,+ \int_0^t \phi(t-t') \left[ \int_{-\infty}^{+\infty} w(x-x')\, p(x',t')\, dx' \right] dt' \,, \qquad (2.1)$$

in which the *survival function*

$$\Psi(t) = \int_t^\infty \phi(t')\, dt' \qquad (2.2)$$

denotes the probability that at instant $t$ the particle is still sitting in its starting position $x = 0$. Clearly, (2.1) satisfies the initial condition $p(x,0) = \delta(x)$.
Note that the *special choice*

$$w(x) = \delta(x-1) \qquad (2.3)$$

gives the *pure renewal process*, with position $x(t) = N(t)$, denoting the *counting function*, and with jumps all of length 1 in positive direction happening at the renewal instants.
For many purposes the integral equation (2.1) of CTRW can be easily treated by using the Laplace and Fourier transforms. Writing these as

$$\mathcal{L}\{f(t);s\} = \widetilde{f}(s) := \int_0^\infty \mathrm{e}^{-st}\, f(t)\, dt\,,\ \mathcal{F}\{g(x);\kappa\} = \widehat{g}(\kappa) := \int_{-\infty}^{+\infty} \mathrm{e}^{+i\kappa x}\, g(x)\, dx\,,$$

then in the Laplace-Fourier domain Eq. (2.1) reads

$$\widehat{\widetilde{p}}(\kappa,s) = \frac{1-\widetilde{\phi}(s)}{s} + \widetilde{\phi}(s)\, \widehat{w}(\kappa)\, \widehat{\widetilde{p}}(\kappa,s)\,, \qquad (2.4)$$

Introducing the auxiliary function $H(t)$ (following [41]) such that

$$\widetilde{H}(s) = \frac{1-\widetilde{\phi}(s)}{s\,\widetilde{\phi}(s)} = \frac{\widetilde{\Psi}(s)}{\widetilde{\phi}(s)}\,, \quad \text{hence} \quad \widetilde{\phi}(s) = \frac{1}{1+s\widetilde{H}(s)}\,, \qquad (2.5)$$

we get the equivalent equation

$$\widetilde{H}(s)\, \left[ s\widehat{\widetilde{p}}(\kappa,s) - 1 \right] = \left[ \widehat{w}(\kappa) - 1 \right] \widehat{\widetilde{p}}(\kappa,s)\,. \qquad (2.6)$$

Turning on the time-space domain by inversion we arrive at the generalized Kolmogorov-Feller equation.

$$\int_0^t H(t-t')\,\frac{\partial}{\partial t'}p(x,t')\,dt' = -p(x,t) + \int_{-\infty}^{+\infty} w(x-x')\,p(x',t)\,dx', \tag{2.7}$$

with $p(x,0)=\delta(x)$, where $H(t)$ acts as a *memory function*
*Special choices of the memory function* are:

$$(i) \quad H(t)=\delta(t) \quad \text{corresponding to} \quad \widetilde{H}(s)=1\,, \tag{2.8}$$

that gives the *exponential waiting time* with

$$\widetilde{\phi}(s) = \frac{1}{1+s}, \quad \phi(t)=\Psi(t)=\mathrm{e}^{-t}\,. \tag{2.9}$$

In this case we obtain in the Fourier- Laplace domain

$$s\widehat{\widetilde{p}}(\kappa,s) - 1 = [\widehat{w}(\kappa)-1]\,\widehat{\widetilde{p}}(\kappa,s)\,, \tag{2.10}$$

and in the space-time domain the *classical Kolmogorov-Feller equation*

$$\frac{\partial}{\partial t}p(x,t) = -p(x,t) + \int_{-\infty}^{+\infty} w(x-x')\,p(x',t)\,dx'\,, \quad p(x,0)=\delta(x)\,. \tag{2.11}$$

$$(ii) \quad H(t)=\frac{t^{-\beta}}{\Gamma(1-\beta)}\,,\ 0<\beta<1\,, \quad \text{corresponding to} \quad \widetilde{H}(s)=\frac{1}{s^{1-\beta}} \tag{2.12}$$

that gives the *Mittag-Leffler waiting time* with

$$\widetilde{\phi}(s)=\frac{1}{1+s^\beta}, \quad \phi(t)=-\frac{d}{dt}E_\beta(-t^\beta), \quad \Psi(t)=E_\beta(-t^\beta)\,. \tag{2.13}$$

In this case we obtain in the Fourier-Laplace domain

$$s^{\beta-1}\left[s\widehat{\widetilde{p}}(\kappa,s)-1\right] = [\widehat{w}(\kappa)-1]\,\widehat{\widetilde{p}}(\kappa,s)\,, \tag{2.14}$$

and in the space-time domain the *time fractional Kolmogorov-Feller equation*

$${}_tD_*^\beta\, p(x,t) = -p(x,t) + \int_{-\infty}^{+\infty} w(x-x')\,p(x',t)\,dx'\,, \quad p(x,0)=\delta(x)\,, \tag{2.15}$$

where ${}_tD_*^\beta$ denotes the fractional derivative of of order $\beta$ in the Caputo sense, see Appendix B.
The time fractional Kolmogorov-Feller equation can be also expressed via the Riemann-Liouville fractional derivative ${}_tD^{1-\beta}$, see again Appendix B, that is

$$\frac{\partial}{\partial t}p(x,t) = {}_tD^{1-\beta}\left[-p(x,t) + \int_{-\infty}^{+\infty} w(x-x')\,p(x',t)\,dx'\right]\,, \quad p(x,0)=\delta(x)\,. \tag{2.16}$$

The equivalence of the two forms (2.15) and (2.16) is easily proved in the Fourier-Laplace domain by multiplying both sides of Eq. (2.14) for the factor $s^{1-\beta}$.

We note that the choice $(i)$ may be considered as a limit of the choice $(ii)$ as $\beta = 1$. In fact, in this limit we find $\widetilde{H}(s) \equiv 1$ so $H(t) = t^{-1}/\Gamma(0) \equiv \delta(t)$ (according to a formal representation of the Dirac generalized function [13]), so that Eqs. (2.6)-(2.7) reduce to (2.10)-(2.11), respectively. In this case the order of the Caputo derivative reduces to 1 and that of the R-L derivative to 0, whereas the Mittag-Leffler waiting-time law reduces to the exponential.

# 3 Manipulations: rescaling and respeeding

We now consider two types of manipulations on the CTRW by acting on its governing equation (2.7) in its Laplace-Fourier representation (2.6).
(A) **rescaling** the waiting time, hence the whole time axis.
(B) **respeeding** the process.

(A) means change of the unit of time (measurement). We replace the random waiting time $T$ by a waiting time $\tau T$, with the positive *rescaling factor* $\tau$. Our idea is to take $0 < \tau \ll 1$ in order to bring into near sight the distant future. In a moderate span of time we will so have a large number of jump events. For $\tau > 0$ we get the rescaled waiting time density

$$\phi_\tau(t) = \phi(t/\tau)/\tau\,, \quad \text{hence} \quad \widetilde{\phi}_\tau(s) = \widetilde{\phi}(\tau s)\,. \tag{3.1}$$

By decorating also the density $p$ with an index $\tau$ we obtain the rescaled integral equation of the CTRW in the Laplace-Fourier domain as

$$\widetilde{H}_\tau(s)\,\left[s\widehat{\widetilde{p}}_\tau(\kappa,s) - 1\right] = \left[\widehat{w}(\kappa) - 1\right]\,\widehat{\widetilde{p}}_\tau(\kappa,s)\,, \tag{3.2}$$

where, according to (2.5),

$$\widetilde{H}_\tau(s) = \frac{1-\widetilde{\phi}(\tau s)}{s\,\widetilde{\phi}(\tau s)}\,. \tag{3.3}$$

(B) means multiplying the quantity $\dfrac{\partial}{\partial t}p(x,t)$ by a factor $1/a$, where $a > 0$ is the *respeeding factor*: $a > 1$ means *acceleration*, $0 < a < 1$ means *deceleration*. In the Laplace-Fourier representation this means multiplying the RHS of Eq. (2.6) by the factor $a$ since the expression $\left[s\widehat{\widetilde{p}}(\kappa,s) - 1\right]$ corresponds to $\dfrac{\partial}{\partial t}p(x,t)$.

We now chose to consider the procedures of rescaling and respeeding in their combination so the equation in the transformed domain of the rescaled and respeeded process has the form

$$\widetilde{H}_\tau(s)\,\left[s\widehat{\widetilde{p}}_{\tau,a}(\kappa,s) - 1\right] = a\,\left[\widehat{w}(\kappa) - 1\right]\,\widehat{\widetilde{p}}_{\tau,a}(\kappa,s)\,, \tag{3.4}$$

Clearly, the two manipulations can be discussed separately: the choice $\{\tau > 0,\ a = 1\}$ means *pure rescaling*, the choice $\{\tau = 1,\ a > 0\}$ means *pure respeeding* of the original process. In the special case $\tau = 1$ we only respeed the original system; if $0 < \tau \ll 1$ we can counteract the compression effected by rescaling to again obtain a moderate number of events in a moderate span of time by respeeding (decelerating) with $0 < a \ll 1$. These vague notions will become clear as soon as we consider power law waiting times.

Defining now

$$\widetilde{H}_{\tau,a}(s) := \frac{\widetilde{H}_\tau(s)}{a} = \frac{1-\widetilde{\phi}(\tau s)}{as\,\widetilde{\phi}(\tau s)}\,. \tag{3.5}$$

we finally get

$$\widetilde{H}_{\tau,a}(s)\,\left[s\widehat{\widetilde{p}}_{\tau,a}(\kappa,s)-1\right] = \left[\widehat{w}(\kappa)-1\right]\,\widehat{\widetilde{p}}_{\tau,a}(\kappa,s)\,. \tag{3.6}$$

What is the combined effect of rescaling and respeeding on the waiting time density? In analogy to (2.6) and taking account of (3.5) we find

$$\widetilde{\phi}_{\tau,a}(s) = \frac{1}{1+s\widetilde{H}_{\tau,a}(s)} = \frac{1}{1+s\dfrac{1-\widetilde{\phi}(\tau s)}{as\,\widetilde{\phi}(\tau s)}}\,, \tag{3.7}$$

and so, for the deformation of the waiting time density, the *essential formula*

$$\widetilde{\phi}_{\tau,a}(s) = \frac{a\,\widetilde{\phi}(\tau s)}{1-(1-a)\widetilde{\phi}(\tau s)}\,. \tag{3.8}$$

**Remark**: The formula (3.8) has the same structure as the thinning formula (A.5) in the Appendix A (just devoted to the thinning theory) by identifying $a$ with $q$. In both problems we have a rescaling process defined by a time scale $\tau$, and we send the relevant factors $\tau$, $a$ and $q$ to zero under a proper relationship. However in the thinning theory the relevant independent parameter going to 0 is that of thinning (actually respeeding) whereas in the present problem it is the rescaling parameter $\tau$.

# 4 Power laws and asymptotic universality of the Mittag-Leffler waiting time density

We have essentially two different situations for the waiting time distribution according to its first moment (the expectation value) being finite or infinite. In other words we assume for the waiting time *pdf* $\phi(t)$ either

$$\rho := \int_0^\infty t'\,\phi(t')\,dt' < \infty\,, \quad \text{labelled as } \beta = 1\,, \tag{4.1}$$

or

$$\phi(t) \sim c\,t^{-(\beta+1)} \text{ for } t\to\infty \quad \text{hence } \; \Psi(t) \sim \frac{c}{\beta}\,t^{-\beta}\,,\; 0<\beta<1\,,\; c>0\,. \tag{4.2}$$

For convenience we have dispensed in (4.2) with decorating by a slowly varying function at infinity[2] the asymptotic power law. Then, by the standard Tauberian theory (see [12, 64]) the above conditions (4.1)-(4.2) mean in the Laplace domain the (comprehensive) asymptotic form

$$\widetilde{\phi}(s) = 1 - \lambda s^\beta + o(s^\beta) \quad \text{for} \quad s \to 0^+ \,, \quad 0 < \beta \le 1 \,, \tag{4.3}$$

where we have

$$\lambda = \rho \,, \quad \text{if} \quad \beta = 1 \,; \quad \lambda = c\Gamma(-\beta) = \frac{c}{\Gamma(\beta+1)} \, \frac{\pi}{\sin(\beta\pi)} \,, \quad \text{if} \quad 0 < \beta < 1 \,. \tag{4.4}$$

Then, *fixing* $s$ as required by the continuity theorem of probability theory for Laplace transforms, taking

$$a = \lambda \tau^\beta \,, \tag{4.5}$$

and *sending* $\tau$ *to zero*, we obtain in the limit the Mittag-Leffler waiting time law. In fact, Eqs. (3.8) and (4.3) imply as $\tau \to 0$

$$\widetilde{\phi}_{\tau,\lambda\tau^\beta}(s) = \frac{\lambda\tau^\beta \left[1 - \lambda\tau^\beta s^\beta + o(\tau^\beta s^\beta)\right]}{1 - (1 - \lambda\tau^\beta)\left[1 - \lambda\tau^\beta s^\beta + o(\tau^\beta s^\beta)\right]} \to \frac{1}{1+s^\beta} \,, \; 0 < \beta \le 1. \tag{4.6}$$

This formula expresses *the asymptotic universality of the Mittag-Leffler waiting time law* that includes the exponential law for $\beta = 1$. It can easily be generalized to the case of power laws decorated with slowly varying functions, thereby using the Tauberian theory by Karamata (see again [12, 64]).

**Comment:** The formula (4.6) says that our general power law waiting time density is gradually deformed into the Mittag-Leffler waiting time density as $\tau$ tends to zero.

**Exercises:** exhibiting the distinguished character of the Mittag-Leffler waiting-time density:
Consider the Mittag-Leffler waiting time law given in (2.12)

$$\widetilde{\phi}^{ML}(s) = \frac{1}{1+s^\beta} \,, \quad \phi^{ML}(t) = -\frac{d}{dt} E_\beta(-t^\beta) \,, \; 0 < \beta \le 1 \,. \tag{4.7}$$

Prove the identity

$$\widetilde{\phi}^{ML}_{\tau,a}(s) = \widetilde{\phi}^{ML}(\tau s / a^{1/\beta}) \quad \text{for all} \quad \tau > 0, a > 0 \,. \tag{4.8}$$

Note that (4.6) states the *self-similarity* of the combined operation *rescaling-respeeding* for the Mittag-Leffler waiting time density. In fact, (4.8) implies $\phi^{ML}_{\tau,a}(t) = \phi^{ML}(t/c)/c$ with $c = \tau/a^{1/\beta}$, which means replacing the random waiting time $T^{ML}$ by $c\,T^{ML}$. As a consequences, choosing $a = \tau^\beta$ we have

$$\widetilde{\phi}^{ML}_{\tau,\tau^\beta}(s) = \widetilde{\phi}^{ML}(s) \quad \text{for all} \quad \tau > 0 \,. \tag{4.9}$$

[2]**Definition:** We call a (measurable) positive function $a(y)$, defined in a right neighbourhood of zero, *slowly varying at zero* if $a(cy)/a(y) \to 1$ with $y \to 0$ for every $c > 0$. We call a (measurable) positive function $b(y)$, defined in a neighbourhood of infinity, *slowly varying at infinity* if $b(cy)/b(y) \to 1$ with $y \to \infty$ for every $c > 0$. A standard example of slow varying function at zero and at infinity is $(\log y)^\gamma$, with $\gamma \in \mathbf{R}$.

Hence *the Mittag-Leffler waiting time law is invariant against combined rescaling with* $\tau$ *and respeeding with* $a = \tau^\beta$. Obviously (4.6) we can say that it is a $\tau \to 0$ attractor for any power law waiting time (4.2) under simultaneous rescaling with $\tau$ and respeeding with $a = \lambda \tau^\beta$.

**Remark**: This attraction property of the Mittag-Leffler probability distribution with respect to power law waiting times (with $0 < \beta \le 1$) is a kind of analogy to the attraction of sums of power law jump distributions by stable distributions.

# 5 Passage to the diffusion limit in space

We have again two different situations for the jump-width distribution according to its second moment being finite or infinite. In other words we assume for the jump-width *pdf* $w(x)$ (assumed for simplicity to be symmetric: $w(x) = w(-x)$) either

$$\sigma^2 := \int_{-\infty}^{+\infty} x^2 \, w(x) \, dx < \infty \,, \quad \text{labelled as } \alpha = 2 \,, \tag{5.1}$$

or

$$w(x) \sim b \, |x|^{-(\alpha+1)} \quad \text{for} \quad |x| \to \infty \,, \quad 0 < \alpha < 2, \quad b > 0 \,. \tag{5.2}$$

Then we have the asymptotic relation

$$\widehat{w}(\kappa) = 1 - \mu \, |\kappa|^\alpha + o(|\kappa|^\alpha) \quad \text{for} \quad \kappa \to 0 \,, \tag{5.3}$$

where

$$\mu = \frac{\sigma^2}{2} \quad \text{if} \quad \alpha = 2, \qquad \mu = \frac{b \, \pi}{\Gamma(\alpha+1) \, \sin{(\alpha\pi/2)}} \quad \text{if} \quad 0 < \alpha < 2, \tag{5.4}$$

As before we dispense with the possible decoration of the relevant power law by a slowly varying function.

By another respeeding, in fact an acceleration, we can pass over to time-space-fractional diffusion processes. For this we have *three choices*:
(a) and (b): diffusion limit in space only, (a) for general waiting time, (b) for ML waiting time,
(c) joint limit in time and space (with power laws in both) with scaling relation.
Note hat (b) is just a special case of (a) but of particular relevance (as we shall see).

In all three cases we rescale the jump density by a factor $h > 0$, replacing the random jumps $X$ by $hX$. This means changing the unit of measurement in space, with $0 < h \ll 1$ so bringing into near sight the far-away space. We get the rescaled jump density as $w_h(x) = w(x/h)/h$, corresponding to $\widehat{w}_h(\kappa) = \widehat{w}_(h\kappa)$.

**Choice (a): diffusion limit in space only, with a general waiting-time law**.
Starting from the Eq. (2.6), the Laplace-Fourier representation of the CTRW equation, without special assumption on the waiting-time density, we fix the Fourier variable $\kappa$ and accelerate the spatially rescaled process by the respeeding factor $1/(\mu h^\alpha)$, arriving at the equation (using $q_h$ as new dependent variable)

$$\widetilde{H}(s) \, \left[ s \widehat{\widetilde{q}}_h(\kappa, s) - 1 \right] = \frac{\widehat{w}(h\kappa) - 1}{\mu h^\alpha} \, \widehat{\widetilde{q}}_h(\kappa, s) \,. \tag{5.5}$$

Then, *fixing* $\kappa$ as required by the continuity theorem of probability theory for Fourier transforms, and *sending* $h$ *to zero* we get, noting that $[\widehat{w}(h\kappa)-1]/(\mu h^\alpha) \to -|\kappa|^\alpha$, and writing $u$ in place of $q_0$,

$$\widetilde{H}(s)\,\left[s\widehat{\widetilde{u}}(\kappa,s)-1\right] = -|\kappa|^\alpha\,\widehat{\widetilde{u}}(\kappa,s)\,, \tag{5.6}$$

where we still have, consistently with (2.5),

$$\widetilde{H}(s) = \frac{1-\widetilde{\phi}(s)}{s\,\widetilde{\phi}(s)} = \frac{\widetilde{\Psi}(s)}{\widetilde{\phi}(s)}\,,$$

being $\phi(t)$ the original waiting time density. In physical space-time we have the integro-pseudo-differential equation

$$\int_0^t H(t-t')\,\frac{\partial}{\partial t'}u(x,t')\,dt' \;=\; {}_xD_0^\alpha\,u(x,t)\,,\quad 0<\alpha\le 2\,, \tag{5.7}$$

with $-|\kappa|^\alpha$ as the symbol of the Riesz pseudo-differential operator ${}_xD_0^\alpha$ usually referred to as the Riesz fractional derivative of order $\alpha$, see Appendix C.

**Comments**: By this rescaling and acceleration the jumps become smaller and smaller, their number in a given span of time larger and larger, the waiting times between jumps smaller and smaller. In the limit there are no waiting times anymore, the original waiting time density $\phi(t)$ is now only spiritual, but still determines via $H(t)$ the memory of the process. Eq. (5.7) offers a great variety of diffusion processes with memory depending on the choice of the function $H(t)$.

**Choice (b): diffusion limit in space only, with a Mittag-Leffler waiting-time law**.

We now choose in Eq. (5.7)

$$H(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}\,,\quad \text{so}\quad \widetilde{H}(s) = 1/s^{(1-\beta)}$$

corresponding to the Mittag-Leffler waiting-time law

$$\phi^{ML}(t) = -\frac{d}{dt}E_\beta(-t^\beta)\,,\quad 0<\beta\le 1\,,$$

consistently with Eqs. (2.11)-(2.12) and with the time-fractional Kolmogorov-Feller equation (2.13). As a consequence of our limit we so arrive immediately at the *time-space fractional diffusion equation*

$${}_tD_*^\beta\,u(x,t) \;=\; {}_xD_0^\alpha\,u(x,t)\,,\quad u(x,0)=\delta(x)\,. \tag{5.8}$$

**Choice (c): diffusion limit in time and in space**.

Assuming the behaviour for the waiting-time density as in Eqs. (4.1)-(4.2), and for the jump-width density as in Eqs. (5.1)-(5.2) rescaling as described the waiting times and the jumps by factors $\tau$ and $h$, starting from (3.2), decelerating by a factor $lambda\,\tau^{\beta}$ in time, then accelerating for space by a factor $1/(\mu h^{\alpha})$, we obtain (compare to Section 3 case (B)), fixing $s$ and $\kappa$ and setting, for convenience

$$a(\tau, h) = \frac{\lambda \tau^{\beta}}{\mu h^{\alpha}}\,, \tag{5.9}$$

$$\widetilde{H}_{\tau}(s)\,\left[s\widehat{\widetilde{p}}_{\tau,a(\tau,h)}(\kappa, s) - 1\right] = a(\tau, h)\,\left[\widehat{w}(h\kappa) - 1\right]\,\widehat{\widetilde{p}}_{\tau,a(\tau,h)}(\kappa, s)\,, \tag{5.10}$$

with

$$\widetilde{H}_{\tau}(s) = \frac{1 - \widetilde{\phi}(\tau s)}{s\,\widetilde{\phi}(\tau s)} \sim \lambda \tau^{\beta} s^{\beta - 1}\,. \tag{5.11}$$

Fixing $a(\tau, h)$ to the constant value 1, then introducing the relationship

$$\frac{\lambda \tau^{\beta}}{\mu h^{\alpha}} \equiv 1\,, \tag{5.12}$$

between the rescaling of time and space, we get

$$\widetilde{H}_{\tau}(s) \sim \lambda \tau^{\beta}\, s^{\beta - 1}\,, \quad \text{for} \quad \tau \to 0\,. \tag{5.13}$$

Because of

$$\frac{\widehat{w}(h\kappa) - 1}{\mu h^{\alpha}} \to -|\kappa|^{\alpha}\,, \quad \text{for} \quad h \to 0\,, \tag{5.14}$$

we finally get the limiting equation

$$s^{\beta - 1}\,\left[s\widehat{\widetilde{u}}(\kappa, s) - 1\right] = -|\kappa|^{\alpha}\,\widehat{\widetilde{u}}(\kappa, s)\,, \tag{5.15}$$

corresponding to Eq. (5.8), the time-space fractional diffusion equation.

**Remark**: The Mittag-Leffler waiting time (choice (b)), obeying the power law asymptotics (4.2) with $\lambda = 1$ led directly to the time-space fractional diffusion equation (5.8), without requirement of rescaling and deceleration in time, and with these procedures we arrive likewise at (5.8). This strange fact is caused by the invariance of the Mittag-Leffler density to the combined effects of rescaling by $\tau$ and deceleration by $\lambda \tau^{\beta}$, expressed in equation (4.9).

**Remark**: The combined passage of $\tau$ and $h$ towards zero can be split by keeping first one of these parameters fixed letting the other tend to zero, then letting the other parameter tend to zero. Under our power law assumptions we an transform (2.7), the basic integral equation of CTRW, into Eq. (2.15) (time-fractional CTRW) by rescaling-respeeding manipulation only in the time-variable, and then by rescaling in space followed by an acceleration into (5.8), the space-time fractional diffusion equation. Or we can transform (2.7) by rescaling in space followed by an acceleration into Eq. (5.7) (general space fractional diffusion with memory), and then by by rescaling-respeeding in the time variable arrive at (5.8).

# 6 The time-fractional drift process

It is instructive to study the spatial transition to the diffusion limit for the Mittag-Leffler renewal process. As said in Section 1 the pure renewal process is obtained by choosing $w(x) = \delta(x-1)$ as the jump width density. For the Mittag-Leffler renewal process we must take (4.5)

$$\phi(t) = \phi^{ML}(t) = -\frac{d}{dt} E_\beta(-t^\beta)\,, \quad 0 < \beta \le 1\,.$$

We have $\widetilde{H}(s) = s^{\beta-1}$, $\widehat{w}(\kappa) = \mathrm{e}^{i\kappa}$, hence

$$s^{\beta-1}\left[s\widehat{\widetilde{p}}(\kappa,s) - 1\right] = \left(\mathrm{e}^{i\kappa} - 1\right)\widehat{\widetilde{p}}(\kappa,s)\,. \tag{6.1}$$

Rescaling in space by a factor $h$ and accelerating (because of $w(\kappa) = \mathrm{e}^{i\kappa} = 1 + i\kappa + o(\kappa)$, for $\kappa \to 0$) this pure renewal process by the factor $1/h$ we get a process

$$s^{\beta-1}\left[s\widehat{\widetilde{q}}_h(\kappa,s) - 1\right] = \frac{1}{h}\left(\mathrm{e}^{ih\kappa} - 1\right)\widehat{\widetilde{q}}_h(\kappa,s)\,,$$

which as $h \to 0$ and $\kappa$ fixed gives

$$s^{\beta-1}\left[s\,\widehat{\widetilde{u}}(\kappa,s) - 1\right] = i\kappa\,\widehat{\widetilde{u}}(\kappa,s)\,, \tag{6.2}$$

which implies

$$\widehat{\widetilde{u}}(\kappa,s) = \frac{s^{\beta-1}}{s^\beta - i\kappa}\,. \tag{6.3}$$

We note that Eq. (6.2) corresponds to the time-fractional drift equation

$${}_tD_*^\beta\, u(x,t) \;=\; -\frac{\partial}{\partial x} u(x,t)\,, \quad u(x,0) = \delta(x)\,, \quad x \in \mathbf{R}\,,\; t \in \mathbf{R}^+\,. \tag{6.4}$$

By using the known scaling rules for the Fourier and Laplace transforms,

$$f(ax) \overset{\mathcal{F}}{\leftrightarrow} a^{-1}\,\widehat{f}(\kappa/a)\,, \quad a > 0\,, \qquad f(bt) \overset{\mathcal{L}}{\leftrightarrow} b^{-1}\,\widetilde{f}(s/b)\,, \quad b > 0\,,$$

we infer directly from (6.3) (thus without inverting the two transforms) the following *scaling property* of the (fundamental) solution

$$u(ax\,,\,bt) = b^{-\beta} u(ax/b^\beta\,,\,t)\,.$$

Consequently, introducing the *similarity variable* $x/t^\beta$, we can write

$$u(x,t) = t^{-\beta}\, U(x/t^\beta)\,, \tag{6.5}$$

where $U(x) \equiv u(x,1)$.

To determine the solution in the space-time domain we can follow two alternative strategies related to the different order in carrying out the inversion of the Fourier-Laplace transforms in (6.3). Indeed we can

(S1) : invert the Fourier transforms getting $\widetilde{u}(x,s)$, and then invert this Laplace transform,
(S2) : invert the Laplace transform getting $\widehat{u}(\kappa,t)$, and then invert these Fourier transforms.

*Strategy (S1):* Recalling the Fourier transform pair,

$$\frac{a}{b-i\kappa} \stackrel{\mathcal{F}}{\leftrightarrow} a\,\mathrm{e}^{-xb}\,\Theta(x)\,, \quad b>0\,,$$

where $\Theta(x)$ denotes the unit step Heaviside function, we get

$$\widetilde{u}(x,s) = s^{\beta-1}\,\mathrm{e}^{-xs^{\beta}}\,\Theta(x)\,. \tag{6.6}$$

*Strategy (S2):* Recalling the Laplace transform pair, see e.g. [10, 18],

$$\frac{s^{\beta-1}}{s^{\beta}-c} \stackrel{\mathcal{L}}{\leftrightarrow} E_{\beta}(ct^{\beta})\,, \quad \Re\,(s) > |c|^{1/\beta}\,,$$

we get

$$\widehat{u}(\kappa,t) = E_{\beta}(i\kappa t^{\beta})\,, \tag{6.7}$$

from which

$$u(x,t) = \frac{1}{2\pi} VP \int_{-\infty}^{+\infty} \mathrm{e}^{-i\kappa x}\, E_{\beta}(i\kappa t^{\beta})\, d\kappa\,, \tag{6.8}$$

where $VP$ denotes the Cauchy principal value[3].
In both cases we arrive at the solution as

$$u(x,t) = t^{-\beta}\, M_{\beta}(x/t^{\beta})\,\Theta(x)\,, \tag{6.9}$$

where $M$ denotes the function of Wright type defined in the complex plane

$$M_{\beta}(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\,\Gamma[-\beta n + (1-\beta)]} = \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{(-z)^{n-1}}{(n-1)!}\,\Gamma(\beta n)\,\sin(\pi\beta n)\,. \tag{6.0}$$

For some notes on this function and on its relationships with Mittag-Leffler functions and with extremal stable densities we refer to Appendix D.
We note the relevant particular case $\beta = 1/2$ for which we obtain

$$\beta = 1/2\ : \quad u(x,t) = \frac{1}{\sqrt{\pi t}} \exp\left[-x^2/(4t)\right]\,, \quad x \ge 0\,, \quad t \ge 0\,. \tag{6.11}$$

[3]From [10], Vol. 3, Chapter XVIII on Miscellaneous Functions Section 18.1 equation (7), we know that

$$E_{\beta}(iy) \sim \frac{i}{\Gamma(1-\beta)y} \quad \text{for} \quad y \to \pm\infty$$

that is we see that $E_{\beta}(iy)$ does not tend to zero fast enough for the integral (6.8) to exist as a regular improper Riemann integral. But there should be no problem for existence as a Cauchy principal value integral, like the integral over the real line of $\sin x/x$ or even of $1/x$ which give the value $\pi/2$ or 0, respectively.

In the limiting case $\beta = 1$ we recover the rightward pure drift,

$$\beta = 1 : \quad u(x,t) = \frac{1}{t}\,\delta(x/t - 1) = \delta(x-t)\,, \quad x \ge 0\,, \quad t \ge 0\,. \tag{6.12}$$

An alternative approach is via the *integral formula of subordination* valid for the Cauchy problem of a generic space-time fractional diffusion equation (characterized by the space parameters $\alpha, \theta$ and by the time parameter $\beta$), see [24, 25]. In our case we write

$$u(x,t) = \int_0^\infty f_{\alpha,\theta}(x,t_*)\, g_\beta(t_*,t)\, dt_*\,, \tag{6.13}$$

with

$$f_{\alpha,\theta}(x,t_*) = \delta(t_* - x)\,, \tag{6.14}$$

and

$$g_\beta(t_*,t) = \frac{t}{\beta}\, L_\beta^{-\beta}\left(t\, t_*^{-1/\beta}\right)\, t_*^{-1/\beta-1}\,, \tag{6.15}$$

where $L_\beta^{-\beta}$ denotes the extremal (unilateral) stable density of order $\beta$ and skewness $-\beta$.

There are two processes involved. One is the unidirectional motion along the $t_*$ axis representing the operational time. This motion happens in physical time $t$ and the *pdf* for the operational time having value $t_*$ is (as density in $t_*$, evolving in physical time $t$) given by (6.15), being related to the extremal stable density $L_\beta^{-\beta}$. The other process is described by the spatial probability density for sojourn of the particle in point $x$ evolving in operational time $t_*$ given by (6.14), that in this case is a rightward pure drift.

Inserting the expressions (6.14)-(6.15) into the subordination formula (6.13) we get

$$u(x,t) = \frac{t}{\beta}\, L_\beta^{-\beta}\left(t\, x^{-1/\beta}\right)\, x^{-1/\beta-1}\,. \tag{6.16}$$

The formula (6.16), expressed in terms of an extremal stable density, is shown to be equivalent to the formula (69), expressed in terms of the $M$-Wright function, if we use the relationship (D.9) with $c = x$ and $r = t$.

We recall the noteworthy property (discussed in [24, 25]) that the subordination formula has the advantage to be used to simulate the particle paths of a (not necessarily Markovian) process governed by a generic space-time fractional diffusion equation in terms of two (necessarily Markovian) *Lévy stable* processes.

# 7 Conclusions

The basic role of the Mittag-Leffler waiting time probability density in time-fractional continuous time random walk (CTRW) has become well known by the fundamental paper of 1995 by Hilfer and Anton [29]. Earlier in the theory of thinning (rarefaction) of a renewal process under power law assumptions, see the 1968 book by Gnedenko and Kovalenko [14], this density had been found as limit density by a combination of thinning followed by rescaling of time and imposing a proper relation between the rescaling factor and the thinning parameter. Likewise one arrives at this law when wanting to construct a certain special class of anomalous random walks, see the 1985 paper by Balakrishnan [1], the anomaly defined by growth of the second moment of the sojourn probability density like a power of time with exponent between 0 and 1. Balakrishnan's paper, having appeared a few years before the fundamental paper of 1989 by Schneider and Wyss [57], is difficult to read as it is written in a style different from the present one, so we will here not go into details. But let it be said that by well-scaled passage to the limit from CTRW (again under suitable power law assumptions in space and time) he obtained the time-space fractional diffusion equation in form of an equivalent integro-differential equation. Unfortunately, Balakrishnan's paper did not find the attention it would have deserved. However, due to the sad fact that the Mittag-Leffler function too long played a rather neglected role in treatises on special functions Balakrishnan as well as Gnedenko and Kovalenko contented themselves with presenting their results only in the Laplace transform domain; they did not identify their limit density as a Mittag-Leffler type function.

Having worked ourselves for some time on questions of well-scaled passage to the diffusion limit from continuous time random walks to fractional diffusion, see [20, 21, 22, 26, 42, 56], we got from the theory of thinning the idea that it should be possible to carry out the passages to the limit separately in space and in time. In time this can be done by a combination of re-scaling time and respeeding the underlying renewal process (formally treating it as a CTRW with unit steps in space). In fact, *thinning* in the sense of Gnedenko and Kovalenko transforms the original renewal process into one that can be obtained from it by a corresponding acceleration. The result of our combination of rescaling and respeeding for a CTRW governed by a given renewal process with a generic power law waiting time law is a time fractional CTRW with a power law memory function. By another rescaling in space (now under power law assumption for the jumps) which can be interpreted as a second respeeding we arrive at the already classical time-space fractional diffusion equation. In this way we shed new light on the large time and wide space behaviour of continuous time random walks. Our trick in considering separately the limiting waiting time law of the renewal process consists in treating such process as a CTRW with positive jumps of size 1 so that its counting number acts as a spatial variable. Then by rescaling this spatial variable in the same way as for a general CTRW we obtain as an interesting side result positively oriented stable processes as limits of renewal processes with power law waiting time.

# Appendix A: The thinning theory

The *thinning* theory for a renewal process has been considered in detail by Gnedenko and Kovalenko [14]. We must note that other Authors, like Szántai [59, 60] speak of *rarefaction* in place of thinning. Let us sketch here the essentials of this theory, thereby in the interest of transparency and easy readability dispensing with the possible decoration of the relevant power law by a slowly varying function.

Denoting by $t_n,\ n = 1, 2, 3, \ldots$ the time instants of events, assuming $0 = t_0 < t_1 < t_2 < t_3 < \ldots$, with *i.i.d.* waiting times $T_1 = t_1$ , $T_k = t_k - t_{k-1}$ for $k \geq 2$, (generically denoted by T), *thinning* (or *rarefaction*) means that for each positive index $k$ a decision is made: the event happening in the instant $t_k$ is deleted with probability $p$ or it is maintained with probability $q = 1 - p$, $0 < q < 1$. This procedure produces a *thinned* or *rarefied* renewal process with fewer events (very few events if $q$ is near zero, the case of particular interest) in a moderate span of time.

To compensate for this loss we want to change the unit of time so that we still have a not very few but still a moderate number of events in a moderate span of time. Such change of the unit of time is equivalent to rescaling the waiting time, multiplying with a positive factor $\tau$ so that we have waiting times $\tau T_1, \tau T_2, \tau T_3, \ldots$, and instants $\tau t_1, \tau t_2, \tau t_3, \ldots$, in the rescaled original process. Our intention is, vaguely speaking, to dispose on $\tau$ in relation to the rarefaction parameter $q$ in such a way that for $q$ near zero in some sense the "average" number of events per unit of time remains unchanged. In an asymptotic sense we will make these considerations precise.

Denoting by $F(t) = P(T \leq t)$ the probability distribution function of the (original) waiting time $T$, by $f(t)$ its density ($f(t)$ is a generalized function generating a probability measure) so that $F(t) = \int_0^t f(t')\, dt'$, and analogously by $F_k(t)$ and $f_k$(t) the distribution and density, respectively, of the sum of $k$ waiting times, we have recursively

$$f_1(t) = f(t)\,, \quad f_k(t) = \int_0^t f_{k-1}(t - t')\, dF(t')\,, \text{ for } k \geq 2\,. \qquad (A.1)$$

Observing that after a maintained event the next one of the original process is kept with probability $q$ but dropped in favour of the second-next with probability $p\,q$ and, generally, $n - 1$ events are dropped in favour of the $n$-th-next with probability $p^{n-1}\, q$, we get for the waiting time density of the thinned process the formula

$$g_q(t) = \sum_{n=1}^{\infty} q\, p^{n-1}\, f_n(t)\,. \qquad (A.2)$$

With the modified waiting time $\tau\, T$ we have

$$P(\tau T \leq t) = P(T \leq t/\tau) = F(t/\tau)\,,$$

hence the density $f(t/\tau)/\tau$, and analogously for the density of the sum of $n$ waiting times $f_n(t/\tau)/\tau$. The density of the waiting time of the rescaled (and thinned) process now turns out as

$$g_{q,\tau}(t) = \sum_{n=1}^{\infty} q\, p^{n-1}\, f_n(t/\tau)/\tau\,. \qquad (A.3)$$

In the Laplace domain we have $\widetilde{f_n}(s) = \left(\widetilde{f}(s)\right)^n$ , hence (using $p - 1 - q$)

$$\widetilde{g_q}(s) = \sum_{n=1}^{\infty} q\, p^{n-1} \left(\widetilde{f}(s)\right)^n = \frac{q\, \widetilde{f}(s)}{1 - (1 - q)\, \widetilde{f}(s)}\,, \qquad (A.4)$$

from which by Laplace inversion we can, in principle, construct the waiting time density of the thinned process. By the rescaling we get

$$\widetilde{g}_{q,\tau}(s) = \sum_{n=1}^{\infty} q\, p^{n-1} \left(\widetilde{f}(\tau s)\right)^n = \frac{q\, \widetilde{f}(\tau s)}{1-(1-q)\, \widetilde{f}(\tau s)} \,. \qquad (A.5)$$

## Infinite thinning under proper rescaling

We are interested in stronger and stronger thinning (*infinite thinning*) and consider to this purpose a scale of processes with the parameters $\tau$ (of *rescaling*) and $q$ (of *thinning*), with $q$ tending to zero *under a scaling relation* $q = q(\tau)$ *yet to be specified.*

We have essentially two different situations for the waiting time distribution according that its expectation value is finite or infinite. In the first case we put

$$\lambda = \int_0^{\infty} t'\, f(t')\, dt' < \infty \,. \qquad (A.6a)$$

In the second case we assume a queue of power law type (dispensing with decorating by a slowly varying function at infinity)

$$\Psi(t) := \int_t^{\infty} f(t')\, dt' \sim \frac{c}{\beta} t^{-\beta} \,,\ t \to \infty \quad \text{if} \quad 0 < \beta < 1 \,, \qquad (A.6b)$$

Then, by the Karamata theory (see [12, 64]) the above conditions mean in the Laplace domain

$$\widetilde{f}(s) = 1 - \lambda\, s^{\beta} + o\left(s^{\beta}\right) \,, \quad \text{for} \quad s \to 0^{+} \,, \qquad (A.7)$$

with a positive coefficient $\lambda$ and $0 < \beta \le 1$. The case $\beta = 1$ obviously corresponds to the situation with finite first moment (A.6a), whereas the cases $0 < \beta < 1$ are related to the power law queue with $c = \lambda\, \Gamma(\beta+1)\, \sin(\beta\pi)/\pi$ .

Now, passing to the limit of $q \to 0$ of infinite thinning under the scaling relation

$$q = \lambda\, \tau^{\beta} \,, \quad 0 < \beta \le 1 \,, \qquad (A.8)$$

between the positive parameters $q$ and $\tau$, the Laplace transform of the rescaled density $\widetilde{g_{q,\tau}}(s)$ in (A.5) of the thinned process tends to

$$\widetilde{g}(s) = \frac{1}{1+s^{\beta}} \,, \qquad (A.9)$$

which corresponds to the Mittag-Leffler density

$$g(t) = -\frac{d}{dt} E_{\beta}(-t^{\beta}) \,. \qquad (A.10)$$

Let us remark that Gnedenko and Kovalenko obtained (A.9) as the Laplace transform of the limiting density but did not identify it as the Laplace transform of a Mittag-Leffler type function.

# Appendix B: The time-fractional derivatives

For a sufficiently well-behaved function $f(t)$ $(t \geq 0)$ we define the *Caputo* time-fractional derivative of order $\beta$ with $0 < \beta < 1$ through

$$\mathcal{L}\left\{ {}_tD_*^\beta\, f(t); s\right\} = s^\beta\, \widetilde{f}(s) - s^{\beta-1}\, f(0^+)\,, \quad f(0^+) := \lim_{t\to 0^+} f(t)\,, \tag{B.1}$$

so that

$${}_tD_*^\beta\, f(t) := \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{f'(\tau)}{(t-\tau)^\beta}\, d\tau\,, \quad 0 < \beta < 1\,. \tag{B.2}$$

Such operator has been referred to as the *Caputo* fractional derivative since it was introduced by Caputo in the late 1960's for modelling the energy dissipation in the rheology of the Earth, see [4, 5]. Soon later this derivative was adopted by Caputo and Mainardi in the framework of the linear theory of viscoelasticity, see [6].

The reader should observe that the *Caputo* fractional derivative differs from the usual *Riemann-Liouville* (R-L) fractional derivative

$${}_tD^\beta\, f(t) := \frac{d}{dt}\left[\frac{1}{\Gamma(1-\beta)} \int_0^t \frac{f(\tau)\, d\tau}{(t-\tau)^\beta}\right]\,, \quad 0 < \beta < 1\,. \tag{B.3}$$

This derivative is defined as the left inverse of the corresponding Riemann Liouville (R-L) fractional integral defined for any $\beta > 0$:

$${}_tJ^\beta\, f(t) := \frac{1}{\Gamma(\beta)} \int_0^t \frac{f(\tau)\, d\tau}{(t-\tau)^{1-\beta}}\,, \tag{B.4}$$

so that ${}_tD^\beta\, {}_tJ^\beta\, f(t) = f(t)$. In other words we note the following definitions in operational terms:

$${}_tD^\beta := {}_tD^1\, {}_tJ^{1-\beta}\,, \quad 0 < \beta < 1\,, \tag{B.5}$$

$${}_tD_*^\beta := {}_tJ^{1-\beta}\, {}_tD^1\,, \quad 0 < \beta < 1\,. \tag{B.6}$$

We note the relationships between the two fractional derivatives (when both of them exist),

$${}_tD_*^\beta\, f(t) = {}_tD^\beta\, \left[f(t) - f(0^+)\right] = {}_tD^\beta\, f(t) - f(0^+)\, \frac{t^{-\beta}}{\Gamma(1-\beta)}\,, \quad 0 < \beta < 1\,. \tag{B.7}$$

As a consequence we can interpret the Caputo derivative as a sort of regularization of the R-L derivative as soon as $f(0^+)$ is finite; in this sense such fractional derivative was independently introduced in 1968 by Dzherbashian and Nersesian [9].

We observe the different behaviour of the two fractional derivatives at the end points of the interval $(0, 1)$, as it can be noted from their operational definitions (B.5), (B.6). In fact, whereas for $\beta \to 1^-$ both derivatives reduce to ${}_tD^1$, due to the fact that the operator ${}_tJ^0 = I$ commutes with ${}_tD^1$, for $\beta \to 0^+$ we have

$$\beta \to 0^+ \Longrightarrow \begin{cases} {}_tD^\beta f(t) \to {}_tD^1\, {}_tJ^1\, f(t) = f(t)\,, \\ {}_tD_*^\beta f(t) \to {}_tJ^1\, {}_tD^1\, f(t) = f(t) - f(0^+)\,. \end{cases} \tag{B.8}$$

The above behaviours have induced us to keep for the Riemann-Liouville derivative the same symbolic notation of the standard derivative of integer order, while for the Caputo derivative to decorate the corresponding symbol with subscript $*$.

For the R-L derivative the Laplace transform reads for $0 < \beta < 1$

$$\mathcal{L}\left\{{}_tD^\beta\, f(t); s\right\} = s^\beta\, \widetilde{f}(s) - g(0^+)\,, \quad g(0^+) = \lim_{t\to 0^+} g(t)\,, \quad g(t) := {}_tJ^{(1-\beta)}\, f(t)\,. \quad (B.9)$$

Thus the rule (B.9) is more cumbersome to be used than (B.1) since it requires the initial value of an extra function $g(t)$ related to the given $f(t)$ through a fractional integral. However, when $f^(0^+)$ is finite we easily recognize $g(0^+) = 0$.

In the limit $\beta \to 1^-$ both derivatives reduce to the derivative of the first order so we recover the corresponding standard formula for the Laplace transform:

$$\mathcal{L}\left\{{}_tD^1\, f(t); s\right\} = s\, \widetilde{f}(s) - f(0^+)\,. \qquad (B.10)$$

We conclude this Appendix noting that in a proper way both derivatives can be generalized for any order $\beta > 1$, see *e.g.* [18, 49].

# Appendix C: The space-fractional derivatives

Let us first recall that a generic linear pseudo-differential operator $A$, acting with respect to the variable $x \in \mathbf{R}$, is defined through its Fourier representation, namely

$$\mathcal{F}\left\{A\, f(x); \kappa\right\} := \int_{-\infty}^{+\infty} \mathrm{e}^{i\kappa x}\, A\, f(x)\, dx = \widehat{A}(\kappa)\, \widehat{f}(\kappa)\,, \quad \kappa \in \mathbf{R} \qquad (C.1)$$

where $\widehat{A}(\kappa)$ is referred to as symbol of $A$, formally given as

$$\widehat{A}(\kappa) = \left(A\, \mathrm{e}^{-i\kappa x}\right)\, \mathrm{e}^{+i\kappa x}\,. \qquad (C.2)$$

The fractional *Riesz* derivative ${}_xD_0^\alpha$ is defined as the pseudo-differential operator with symbol $-|\kappa|^\alpha$. This means that for a sufficiently well-behaved (generalized) function $f(x)$ ($x \in \mathbf{R}$) we have

$$\mathcal{F}\left\{\, {}_xD_0^\alpha\, f(x); \kappa\right\} = -|\kappa|^\alpha\, \widehat{f}(\kappa)\,, \quad \kappa \in \mathbf{R}\,. \qquad (C.3)$$

The symbol of the Riesz fractional derivative is nothing but the logarithm of the characteristic function of the generic symmetric *stable* (in the Lévy sense) probability density, see [11, 12, 53]. Noting $-|\kappa|^\alpha = -(\kappa^2)^{\alpha/2}$, we recognize that

$${}_xD_0^\alpha = -\left(-\frac{d^2}{dx^2}\right)^{\alpha/2}\,. \qquad (C.4)$$

In other words, the Riesz derivative is a symmetric fractional generalization of the second derivative to orders less than 2: following an illuminating notation introduced by Zaslavsky, see *e.g.* [52] it is denoted also as $\frac{d}{d|x|^\alpha}$. In an explicit way, for $0 < \alpha < 2$ the Riesz derivative reads

$$ {}_xD_0^\alpha f(x) = \Gamma(1+\alpha) \frac{\sin(\alpha\pi/2)}{\pi} \int_0^\infty \frac{f(x+\xi) - 2f(x) + f(x-\xi)}{\xi^{1+\alpha}} \, d\xi \,. \qquad (C.5) $$

This operator is referred to as the *Riesz fractional derivative* since it is obtained from the inversion of the fractional integral originally introduced by Marcel Riesz in the late 1940's, known as the *Riesz potential*, see *e.g.* [53]. It is based on a suitable regularization of a hyper-singular integral, according to a method formerly introduced by Marchaud in 1927.

**Remark**: Straightforward generalization to the Riesz-Feller derivative of order $\alpha$ and skewness $\theta$ is possible. Such pseudo-differential operator is denoted by us as

$$ {}_xD_\theta^\alpha \,, \quad \text{with} \quad 0 < \alpha \le 2 \,, \quad \theta \in \mathbf{R}, \quad |\theta| \le \min\{\alpha, 2-\alpha\} \,. \qquad (C.6) $$

For its definition we refer the reader e.g. to [19, 39].

# Appendix D: On the $M$-Wright functions

We recall that the function $M_\beta(z)$ with $0 < \beta < 1$ introduced in the text is an entire function of order $\rho = 1/(1-\beta)$ , defined by the series

$$ M_\beta(z) = \sum_{n=0}^\infty \frac{(-z)^n}{n!\,\Gamma[-\beta n + (1-\beta)]} = \frac{1}{\pi} \sum_{n=1}^\infty \frac{(-z)^{n-1}}{(n-1)!} \Gamma(\beta n) \, \sin(\pi \beta n) \,. \qquad (D.1) $$

In the limiting case $\beta = 1$ we get a formal representation of the Dirac generalized function centred in $z = 1$, i.e. $M_1(z) = \delta(z-1)$ . Particular noteworthy cases are obtained for $\beta = 1/2, 1/3$ in that we obtain

$$ M_{1/2}(z) = \frac{1}{\sqrt{\pi}} \exp\left(-\, z^2/4\right) \,, \qquad (D.2) $$

$$ M_{1/3}(z) = 3^{2/3} \, \mathrm{Ai}\left(z/3^{1/3}\right) \,, \qquad (D.3) $$

where Ai denotes the *Airy function.* In other words we may say that such function provides a generalization of the Gaussian and of the Airy function.

The $M$ function is a special case of the Wright function defined by the series representation, valid in the whole complex plane,

$$ \Phi_{\lambda,\mu}(z) := \sum_{n=0}^\infty \frac{z^n}{n!\,\Gamma(\lambda n + \mu)} \,, \quad \lambda > -1 \,, \quad \mu \in \mathbf{C} \,, \quad z \in \mathbf{C} \,. \qquad (D.4) $$

Indeed, we recognize

$$ M_\beta(z) = \Phi_{-\beta, 1-\beta}(-z) \,, \quad 0 < \beta < 1 \,. \qquad (D.5) $$

Originally, Wright introduced and investigated this function with the restriction $\lambda \geq 0$ in a series of notes starting from 1933 in the framework of the asymptotic theory of partitions. Only later, in 1940, he considered the case $-1 < \lambda < 0$. We note that in the handbook of the Bateman Project [10] (see Vol. 3, Ch. 18), presumably for a misprint, $\lambda$ is restricted to be non negative. For more information on the Wright-type functions and their use in time-fractional diffusion equations the interested reader may consult e.g. [16, 17, 23, 34, 35].

The $M$-Wright function is related to the class of the Mittag-Leffler functions through Laplace transformation, see in [39] Eq (4.26a),

$$M_\beta\,(r/c) \stackrel{\mathcal{L}}{\leftrightarrow} c\,E_\beta(-cs)\,, \quad c > 0\,, \quad r > 0\,; \quad \Re\,(s) > 0\,. \tag{D.6}$$

We note that this Laplace transform pair can be used for (formally) carrying out the Fourier integral (28), by analytic continuation $s \to \pm i\kappa$,

The $M$-Wright function is also related to the special class of functions that represent the unilateral stable densities. For this purpose we recall from [39] that the fundamental solution of the Cauchy problem (with $0 < \beta \leq 1$)

$$\frac{\partial}{\partial t} u(x,t) = {}_x D^{\beta}_{-\beta}\, u(x,t)\,, \quad u(x,0) = \delta(x)\,, \quad x \in \mathbf{R}\,,\; t \in \mathbf{R}^+\,, \tag{D.7}$$

is

$$u(x,t) = t^{-\beta}\, L_\beta^{-\beta}(x/t^\beta)\,. \tag{D.8}$$

where $L_\beta^{-\beta}(x)$ denotes the unilateral extremal stable density (of order $\beta$ and skewness $-\beta$), vanishing for $x < 0$.

We recall from Eq. (4.32) in [39] the relationship

$$\frac{1}{c^{1/\beta}}\, L_\beta^{-\beta}\left(\frac{r}{c^{1/\beta}}\right) = \frac{c\,\beta}{r^{\beta+1}}\, M_\beta\left(\frac{c}{r^\beta}\right)\,, \quad 0 < \beta < 1\,, \quad c > 0\,, \quad r > 0\,. \tag{D.9}$$

We also note the Laplace transform pair,

$$L_\beta^{-\beta}\,(r/c) \stackrel{\mathcal{L}}{\leftrightarrow} c\exp\left[-(cs)^\beta\right]\,,\; 0 < \beta < 1\,,\; c > 0\,,\; r > 0\,;\; \Re\,(s) > 0\,. \tag{D.10}$$

Comparing with (D.6), we can recognize some interrelations between the stretched exponential and Mittag-Leffler function from one side and the unilateral stable density and the $M$-Wright function from the other side, as far as the Laplace transform is concerned.

Relevant cases are obtained in [39] if $\beta = 1/2$ [see (4.7), Lévy Smirnov] and (in the limit) for $\beta = 1$ [see (4.10) pure drift], for which the stable densities read respectively ($r \geq 0$)

$$\beta = 1/2\,, \quad L_{1/2}^{-1/2}(r) = \frac{r^{-3/2}}{2\sqrt{\pi}}\, \mathrm{e}^{-1/(4r)}\,, \tag{D.11}$$

$$\beta = 1\,, \quad L_1^{-1}(r) = \delta(r-1)\,. \tag{D.12}$$

The asymptotic representation as $r \to 0^+$ of the general extremal stable density for $0 < \beta < 1$ reads [see (4.15)]

$$L_\beta^{-\beta}(r) \sim A_1 \, x^{-a_1} \, \mathrm{e}^{-b_1 \, r^{-c_1}} \,, \quad r \to 0^+ \,, \quad A_1 = \left\{ [2\pi(1-\beta)]^{-1} \, \beta^{1/(1-\beta)} \right\}^{1/2} \,,$$
$$a_1 = \frac{2-\beta}{2(1-\beta)} \quad b_1 = (1-\beta)\, \beta^{\beta/(1-\beta)} \,, \quad c_1 = \frac{\beta}{1-\beta} \,. \qquad (D.13)$$

Note that for $\beta = 1/2$ the asymptotic representation (D.13) (that is obtained by the saddle point method) provides the exact solution (D.11) !!!!!

The asymptotic representation as $r \to \infty$ is readily obtained from (D.9) by taking the leading term in (D.1) and reads

$$L_\beta^{-\beta}(r) \sim B_1 \, r^{-(\beta+1)} \,, \quad r \to \infty \,, \quad B_1 = \beta/\Gamma(1-\beta) = \Gamma(1+\beta) \, \sin(\pi\beta)/\pi \,. \qquad (D.14)$$

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