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CONFINED LÉVY FLIGHTS AND RELATED TOPICS

A. Chechkin

Institute for Theoretical Physics of National Science Center
«Kharkov Institute of Physics and Technology», Kharkov, UKRAINE,

In collaboration with

- **V. Gonchar (ITP Kharkov)**
- **J. Klafter (TAU)**
- **R. Metzler (University of Ottawa)**
- **I. Sokolov (HU)**

A. Sliusarenko (Kharkov University), T. Koren (Tel Aviv University)

A. Chechkin, V. Gonchar, J. Klafter and R. Metzler, Fundamentals of Lévy Flight Processes. Adv. Chem. Phys. 133B, 439 (2006).

LÉVY motion or "**LÉVY flights**" (Mandelbrot): paradigm of non-Brownian random motion

Mathematical Foundations:

Brownian Motion	Lévy Motion
1. Central Limit Theorem	1. Generalized Central Limit Theorem
2. Properties of Gaussian probability laws and processes	2. Properties of Lévy stable laws and processes

B.V. Gnedenko, A.N. Kolmogorov, Limit Distributions for Sums of Independent Random Variables (1949).

1. *The prophecy:* "All these distribution laws, called stable, deserve the most serious attention. It is probable that the scope of applied problems in which they play an essential role will become in due course rather wide."
2. *The guidance for those who write books on statistical physics:* "… "normal" convergence to abnormal (that is, different from Gaussian –A. Ch.) stable laws … , undoubtedly, have to be considered in every large textbook, which intends to equip well enough the scientist in the field of statistical physics."

TOPICS

- **free Lévy flights in a semi-infinite domain:**

  Sparre Andersen, method of images, Skorokhod and long leapovers;

- **confined Lévy flights in a potential wells:**

  steep asymptotics, bifurcations and bimodality of the PDFs

- **damped Lévy flights:**

  nonlinear friction $\Rightarrow$ finite energy

- **power-law truncated Lévy flights:**

  how the truncated Cauchy PDF transforms into Gaussian

Why just Lévy stable probability laws ?

Exclusive properties of the Lévy stable laws (Paul LÉVY, 1886 -1971)

1. Solid probabilistic background: Generalized Central Limit Theorem

• occur if evolution of the physical system or the result of experiment is determined by the sum of a large number of random quantities

2. Long power-law asymptotics of the PDFs: for symmetric laws

$$\hat{p}_X(k;\alpha,D) \equiv \langle \exp(ikX) \rangle = \exp\left(-D\,|k|^{\alpha}\right), 0<\alpha\le 2, D>0 \underset{0<\alpha<2}{\Rightarrow} p_X(x,\alpha,D) \underset{x\to\infty}{\propto} \frac{1}{|x|^{1+\alpha}}$$

Lévy index $0<\alpha<2 \Rightarrow \langle x^2 \rangle = \infty$ *"fat tails"*; $\alpha = 2$ *Gaussian;* $\alpha > 2$ *is not the PDF* !

• serve for the description of random processes with large outliers

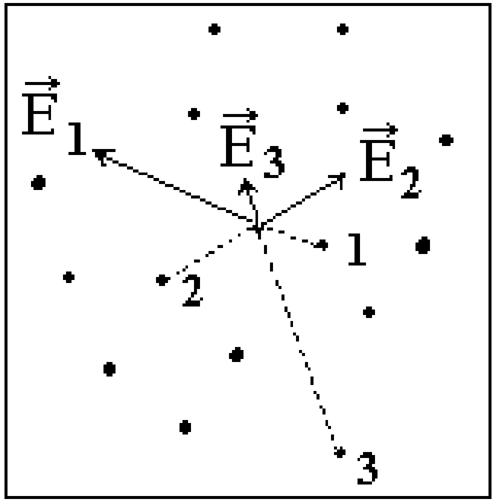
α=2: $p(x,2,D) = \dfrac{1}{\sqrt{4\pi D}}\exp\left(-\dfrac{x^2}{4D}\right)$ (Gauss) $\leftrightarrow$ α=1: $p(x,1,D) = \dfrac{D}{\pi\left(D^2+x^2\right)}$ (Cauchy)

3. Statistical self – similarity: $\Delta L_{\alpha}(\kappa\tau) \overset{d}{=} \kappa^{1/\alpha}\Delta L_{\alpha}(\tau)$

• serve for the description of random fractal processes

Lévy statistics provides a framework for the description of many natural phenomena from a general common point of view

Electric Field Distribution in an Ionized Gas (Holtzmark 1919)
***Application:* spectral lines broadening in plasmas**



$$\vec{E}_i = \frac{e\vec{r}_i}{r_i^3} \qquad n = \frac{N}{V} = const$$

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \ldots + \vec{E}_n$$

$$W\left(\vec{E}\right) = \int_{-\infty}^{\infty} \frac{d\vec{k}}{(2\pi)^3} \widehat{W}\left(\vec{k}\right) e^{-i\vec{k}\vec{E}}$$

$$\widehat{W}\left(\vec{k}\right) = e^{-a\left|\vec{k}\right|^{3/2}} \qquad \widehat{W}\left(\left|\vec{E}\right|\right) \sim \frac{1}{\left|\vec{E}\right|^{5/2}}$$

3 – dimensional
symmetric
stable
distribution
$\alpha = 3/2$

Related examples

- **gravitation field of stars (=3/2)**

- **distribution of temperature in nuclear reactor (=5/3)**

- **distribution of tensions in crystal lattice (=1)**

- **electric fields of dipoles (= 1) and quadrupoles (= 3/4)**

- **velocity field of point vortices in fully developed turbulence (=3/2)**

- **also: asymmetric Levy stable distributions in the first passage theory, single molecule line shape cumulants in glasses …**

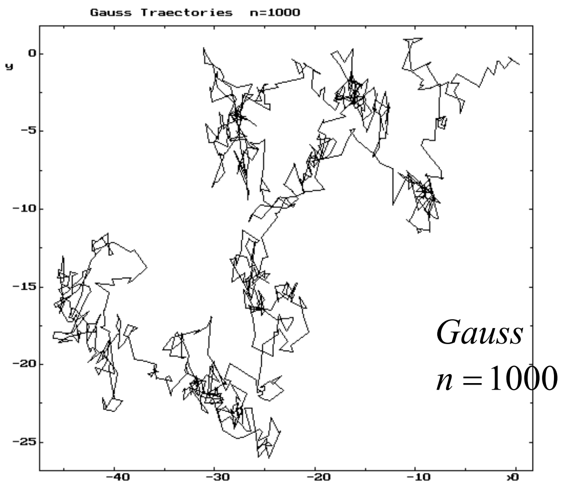
Normal vs Anomalous Diffusion

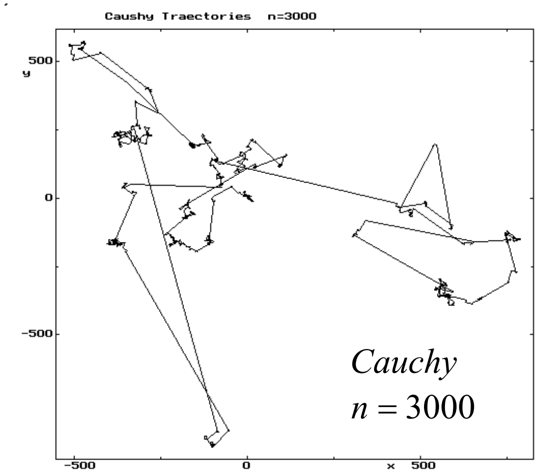
Normal diffusion $\left\langle \left(\vec{R}-\vec{R}_0\right)^2 \right\rangle = c_{\dim} D t \propto t^1$

Bachelier (1900), *Einstein* (1905)

Anomalous diffusion

$$\left\langle \left(\vec{R}-\vec{R}_0\right)^2 \right\rangle \propto t^{\mu},\ \mu \neq 1$$





Superdiffusion

$\mu > 1$

- *turbulent media (gases, fluids, plasmas)*
- *Hamiltonian chaotic systems*
- *deterministic maps*
- *foraging movement*

Subdiffusion

$\mu < 1$

- *turbulent plasmas*
- *transport on fractals*
- *contamimants in underground water*
- *amorphous solids*
- *convective patterns*
- *polymeric systems*
- *deterministic maps*

Impulsive Noises

Modeled with the Lévy Stable Distributions

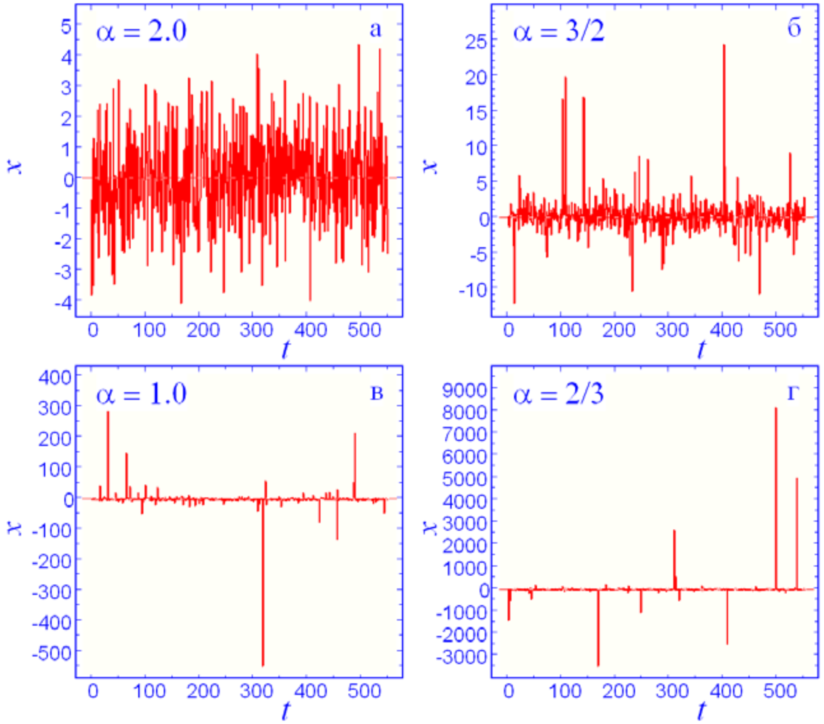
Naturally serve for the description of the processes
with large outliers, far from equilibrium

Examples include :

- **economic stock prices and current exchange rates (1963)**
- **low frequency atmospheric noise**
- **fluorescent lighting systems**
- **biomedical signals**
- **radio and underwater acoustic channels**
- **telecommunications**
- **computer network traffic**
- **stochastic climate dynamics**
- **turbulence in the edge plasmas of thermonuclear devices (Uragan 2M, ADITYA, 2003; Heliotron J, 2005 ...)**

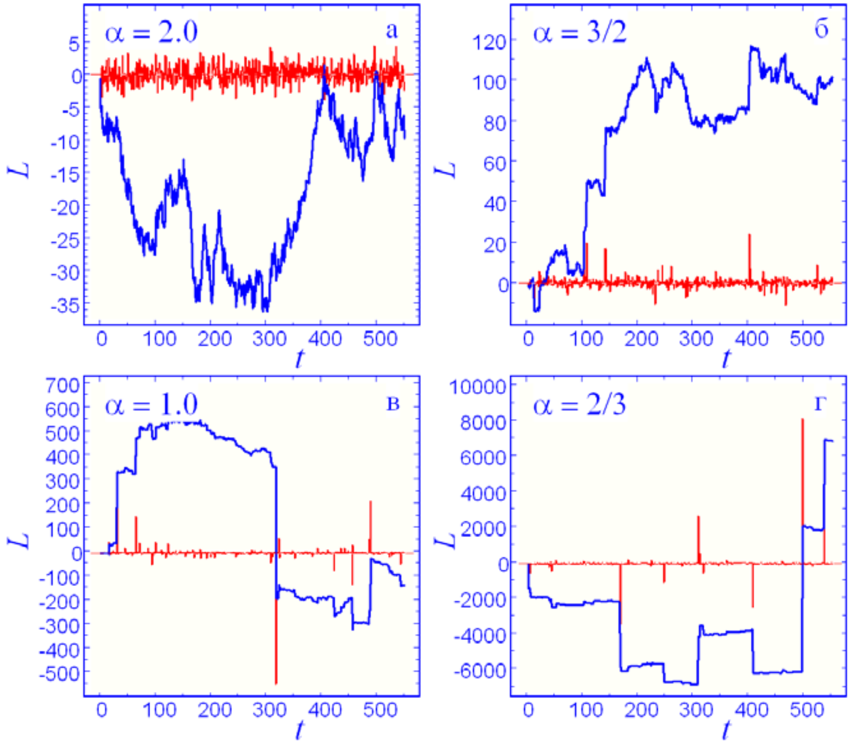
Lévy noises

Lévy index ↓ ⇒ outliers ↑



Lévy motion

Lévy index ↓ ⇒ "flights" become longer



Kinetic equations for the stochastic systems driven by white Lévy noise

(*<u>assumptions</u>: overdamped case, or strong friction limit, 1-dim*, *D = intensity of the noise = const: no Ito-Stratonovich dilemma!*)

Langevin description, $x(t)$

$$\frac{dx}{dt} = -\frac{dU}{dx} + Y_\alpha(t)$$

$U(x)$: potential energy, $Y_\alpha(t)$: white noise, α : the Lévy index

$\alpha = 2$: white Gaussian noise

$0 < \alpha < 2$: white Lévy noise (stationary sequence of independent stationary increments of the Lévy stable process)

Kinetic description, $f(x,t)$

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial x}\left(\frac{dU}{dx} f\right) + D\frac{\partial^\alpha f}{\partial |x|^\alpha}$$

$\partial^\alpha / \partial |x|^\alpha$: **Riesz fractional derivative**: integrodifferential operator

$\alpha = 2$: Fokker - Planck equation (FPE)

$0 < \alpha < 2$: Fractional FPE (FFPE)

Definition of the Riesz fractional derivative

via its Fourier representation

$$\hat{\phi}(k) = \int_{-\infty}^{\infty} dx\, e^{ikx}\phi(x) \quad \overset{FT}{\Leftrightarrow} \quad \phi(x)$$

$$\frac{d^{\alpha}\phi}{d|x|^{\alpha}} \quad \overset{FT}{\Leftrightarrow} \quad -|k|^{\alpha}\hat{\phi}(k)$$

Indeed, $\frac{d^2\phi}{d|x|^2} = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\left(-k^2\right)\hat{\phi}(k)e^{-ikx} = \frac{d^2\phi}{dx^2}$. But: $\frac{d\phi}{d|x|} \overset{FT}{\Leftrightarrow} -|k|\hat{\phi}(k) \neq \frac{d\phi}{dx} \overset{FT}{\Leftrightarrow} -ik\hat{\phi}(k)$!

Examples.

1. $\phi(x) = \frac{1}{1+x^2}$, $\hat{\phi}(k) = \pi \exp(-|k|)$

$$\frac{d^{\alpha}\phi}{d|x|^{\alpha}} = -\int_{-\infty}^{\infty} \frac{dk}{2\pi} \exp(-ikx)|k|^{\alpha}\hat{\phi}(k) == -\frac{\Gamma(\alpha+1)}{\pi\left(1+x^2\right)^{(\alpha+1)/2}}\cos[(\alpha+1)\arctan x]$$

2. $\phi(x) = \exp\left(-x^2\right)$

$$\frac{d^{\alpha}\phi}{d|x|^{\alpha}} = -\frac{2^{\alpha}}{\sqrt{\pi}}\Gamma\left(\frac{\alpha+1}{2}\right) {}_1F_1\left(\frac{\alpha+1}{2};\frac{1}{2};-x^2\right)$$

Fractional diffusion equation, $U = 0$: *Free Lévy flights*

Equation for the PDF: $$\frac{\partial f}{\partial t} = D\frac{\partial^{\alpha} f}{\partial |x|^{\alpha}}\ ,\ D = const,\ f(x,0) = \delta(x)$$

After the Fourier transform : $$\frac{\partial \hat{f}}{\partial t} = -D|k|^{\alpha}\hat{f}\quad ,\quad \hat{f}(k,0) = 1\qquad .$$

Solution in Fourier space: $$\hat{f}(k,t) = \exp\left(-D|k|^{\alpha}t\right):$$

Second moment is infinite for free Lévy flights

Free Lévy flights vs Brownian motion in semi-infinite domain

	Brownian Motion, $\alpha = 2$	Lévy Motion, $0<\alpha<2$
First passage time PDF	$p(t) = \frac{a}{\sqrt{4\pi D t^3}} e^{-a^2/4Dt} \propto t^{-3/2}$	$p(t) \propto t^{-3/2}$

Sparre Andersen universality : consequence of the Skhorokhod's theorem

Method of images	$f_{im}(x,t) = W(x-a,t) - W(x+a,t)$	$p_{im}(t) \propto t^{-1-1/\alpha} \neq t^{-3/2}$

Method of images breaks down for Lévy flights

Leapover: how large is the leap d over the boundary by the Lévy stable process ?	No leapover	$f(d) \propto d^{-1-\alpha/2}$

Leapover PDF decays slower than Lévy flight PDF (consequence of the Skhorokhod's theorem)

Reminding: **Stationary solution of the FPE in external field, $\alpha = 2$**

Equation for the stationary PDF:

$$\frac{d}{dx}\left(\frac{dU}{dx}f\right)+D\frac{d^2 f}{dx^2}=0 \quad . \tag{1}$$

Boltzmann' solution:

$$f(x)=C\exp\left(-\frac{U(x)}{D}\right) \quad , \quad \int_{-\infty}^{\infty} dxf(x)=1 \quad , \tag{2}$$

$$U(x)=\frac{ax^2}{2};\frac{bx^4}{4};\frac{ax^2}{2}+\frac{bx^4}{4};\frac{bx^{2m+2}}{2m+2} \quad ,a>0,\, b>0,\, m=0,1,2,...$$

Two properties of stationary PDF:
1. Unimodality (one hump at the origin).
2. Exponential decay at large values of x.
Qualitatively:
- **the form of stationary PDF is determined by the form of the potential energy;**
- **the role of the Gaussian noise is only to change the width of the PDF.**

Lévy flights in a harmonic potential, $1 \le \alpha < 2$

FFPE for the stationary PDF (dimensionless units):

$$\frac{d}{dx}\left(\frac{dU}{dx} f\right) + \frac{d^{\alpha} f}{d|x|^{\alpha}} = 0 \qquad . \tag{1}$$

$U(x) = \dfrac{x^2}{2}$, (Remind: pass to the Fourier space $\dfrac{d^{\alpha}}{d|x|^{\alpha}} \overset{FT}{\leftrightarrow} -|k|^{\alpha}$) $f(x) \overset{FT}{\leftrightarrow} \hat{f}(k)$

Equation for the characteristic function:

$$\frac{d\hat{f}}{dk} = -\operatorname{sgn}(k)|k|^{\alpha-1}\hat{f}(k) \qquad \hat{f}(k) = \exp\left(-\frac{|k|^{\alpha}}{\alpha}\right)$$: symmetric stable law

Two properties of stationary PDF:

1. Unimodality (one hump at the origin).

2. Slowly decaying tails: $f(|x| \to \infty) \approx \dfrac{C}{|x|^{1+\alpha}}$, $C = \dfrac{\sin(\pi\alpha/2)\Gamma(\alpha)}{\pi}$,

$$\left\langle x^2 \right\rangle = \int_{-\infty}^{\infty} dx\, x^2 f(x) = \infty$$

Harmonic force is not “strong” enough to “confine” Lévy flights

Part I. Confined Lévy flights. Quartic potential well, $U \propto x^4$

Langevin equation: $\frac{dx}{dt} = -x^3 + Y_\alpha(t)$ $\Rightarrow$ Fractional FPE for the stationary PDF:

$$\frac{d}{dx}\left(x^3 f\right) + \frac{d^\alpha f}{d|x|^\alpha} = 0 \quad . \tag{1}$$

$$\left(\text{reminder}: \quad \frac{d^\alpha}{d|x|^\alpha} \overset{FT}{\leftrightarrow} -|k|^\alpha\right) \qquad f(x) \overset{FT}{\leftrightarrow} \hat{f}(k)$$

Equation for the characteristic function:

$$\frac{d^3 \hat{f}}{dk^3} = \mathrm{sgn}(k)|k|^{\alpha-1} \hat{f}(k) \quad , \tag{2}$$

+ normalization + symmetry + boundary conditions …

Confined Lévy flights. Quartic potential, $U \propto x^4$. Cauchy case, $\alpha = 1$

Equation for the characteristic function:

$$\frac{d^3 \hat{f}}{dk^3} = \mathrm{sgn}(k)\hat{f}(k) \qquad \Rightarrow \qquad \hat{f}(k) = \frac{2}{\sqrt{3}} \exp\left(-\frac{|k|}{2}\right) \cos\left(\frac{\sqrt{3}|k|}{2} - \frac{\pi}{6}\right) .$$

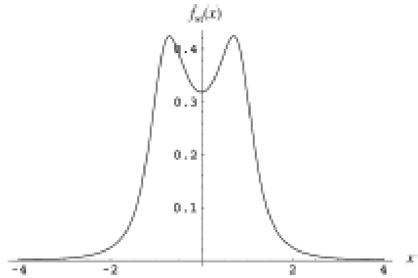
PDF :

$$f(x) = \frac{1}{\pi(1 - x^2 + x^4)} .$$

Two properties of stationary PDF.

1. **Bimodality**: local minimum at $x_{\min} = 0$, two maxima at $x_{\max} = \pm 1/\sqrt{2}$.
2. . **Steep power law** asymptotics with **finite variance**: $f(x) \propto x^{-4} \Rightarrow$

$$\left\langle x^2 \right\rangle = \int_{-\infty}^{\infty} dx \quad x^2 f(x) < \infty$$



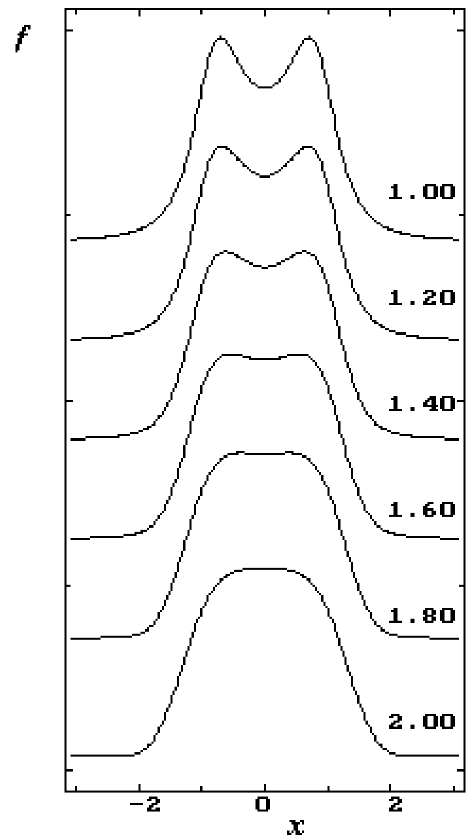
Stationary PDF (35) of the Cauchy-LF in a quartic ($c = 4$) potential. Two global maxima exist at $x_{\max} = \pm\sqrt{1/2}$, and there is a local minimum at the origin.

Confined Lévy flights, $U = x^4$, $1 < \alpha < 2$. Two properties of stationary PDF

1. **Bimodality** (obtained by inverse Fourier transform).

2. **Steep power law asymptotics** (determined by the first non-analytical term in the series for $\hat{f}(k)$): $f(x) \approx \dfrac{C}{|x|^{\alpha+3}}$, $|x| \to \infty$

$\Rightarrow$ finite variance = "confinement" for $1 < \alpha < 2$



Two propositions

Proposition 1. Stationary PDF for the Lévy flights in external field $U = |x|^c$, $c > 2$ is non-unimodal. Proved with the use of the "hypersingular" representation of the Riesz derivative.

Proposition 2. Stationary PDF for the Lévy flights in external field $U = |x|^c$, $c \geq 2$ has power-law asymptotics,

$$f(x) \approx \frac{C_\alpha}{|x|^{\alpha+c-1}} \quad , \quad |x| \to \infty$$

$C_\alpha = \Gamma(\alpha)\pi^{-1}\sin(\pi\alpha/2)$ is a "universal constant", i.e., it does not depend on c.

Critical exponent c_{cr}: $c_{cr} = 4 - \alpha$

$$c < c_{cr} \Rightarrow \left\langle x^2 \right\rangle = \infty$$

$$c > c_{cr} \Rightarrow \left\langle x^2 \right\rangle < \infty \text{ : "confined"}$$

Proved with the use of the representation of the Riesz derivative in terms of left- and right Liouville-Weyl derivatives.

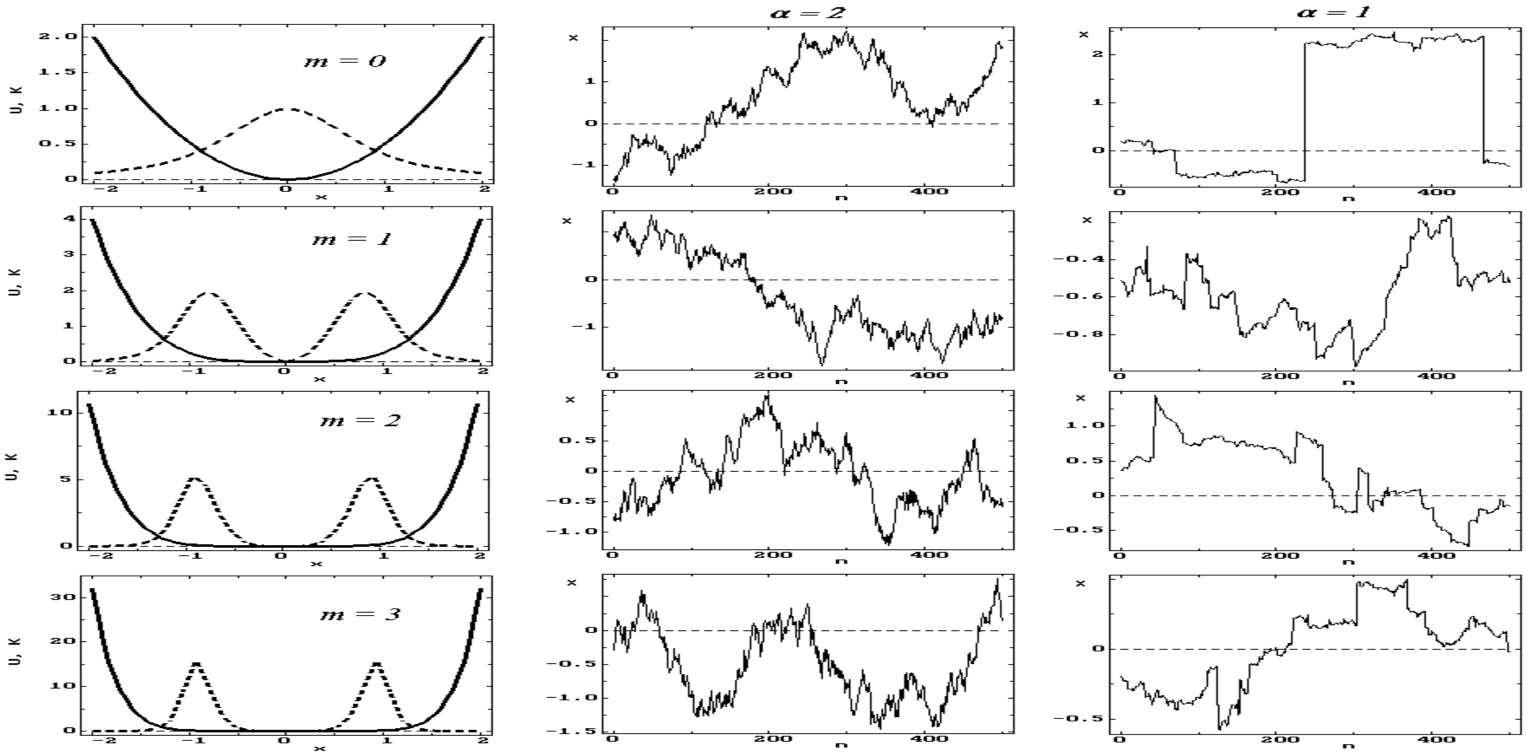
Numerical simulation

Langevin equation with a white Lévy noise,

$$\frac{dx}{dt} = -\frac{dU}{dx} + Y_{\alpha}(t) \ , \qquad U \propto x^{2m+2} \ , \quad m = 0,1,2,...$$

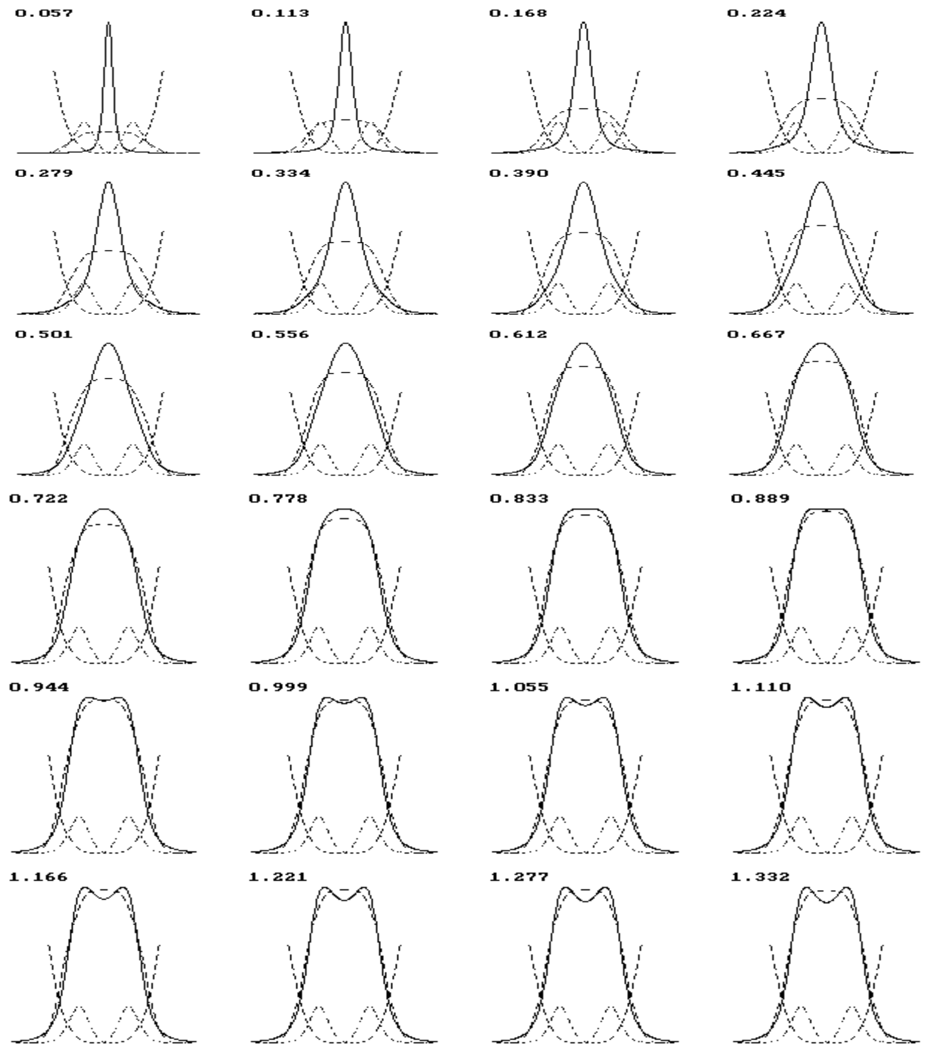
Illustration: Typical sample paths

Potential wells *Brownian motion* *Lévy motion*



$m \uparrow \Rightarrow$ walls are steeper $\Rightarrow$ Lévy flights are shorter $\Rightarrow$ ***confined Lévy flights***

Quartic potential $U \propto x^4$, $\alpha = 1.2$. Time evolution of the PDF.



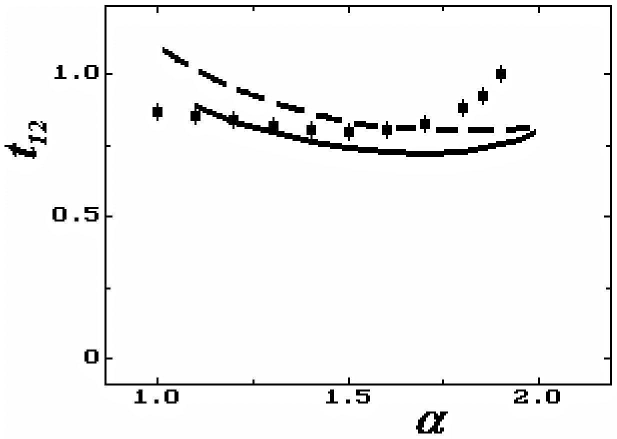
Unimodal-bimodal bifurcation during relaxation in quartic potential well

Bifurcation time t_{12} $\left.\dfrac{\partial^2 f}{\partial x^2}\right|_{x=0,t=t_{12}} = 0$ or, equiv., in terms of charact. funct. $J(t) = \int\limits_0^\infty dk k^2 \hat{f}(k,t) \Rightarrow$

$J(t_{12}) = 0$ (*)

Equation for $\hat{f}$ $\hat{\hat{f}}_1$, $\hat{\hat{f}}_2$ ... $\dfrac{\partial \hat{f}}{\partial t} + |k|^\alpha \hat{f} = \mathbf{U}_k \hat{f}$, (**)

where $\mathbf{U}_k \hat{f} = \int\limits_{-\infty}^{\infty} dx \exp(ikx) \dfrac{\partial}{\partial x}\left(\dfrac{dU}{dx} f\right) = -ik \int\limits_{-\infty}^{\infty} dx \exp(ikx) \operatorname{sgn} x \cdot |x|^{c-1} f(x,t)$

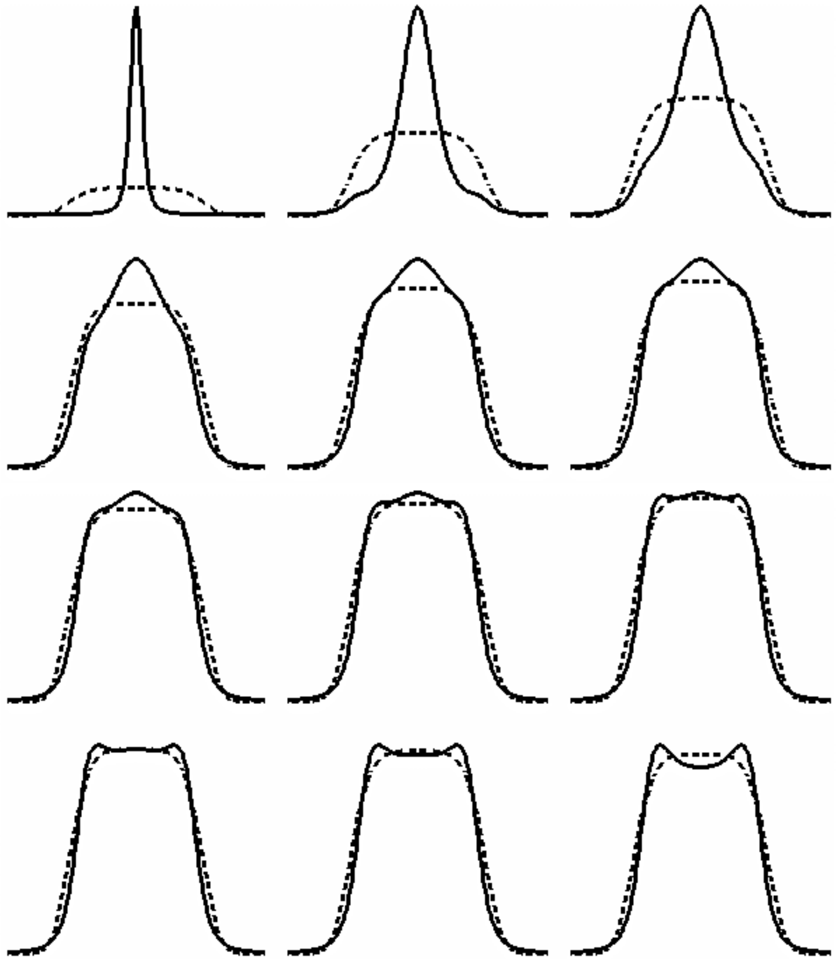


• "exact" bifurcation time (numerical solution of the FFPE)

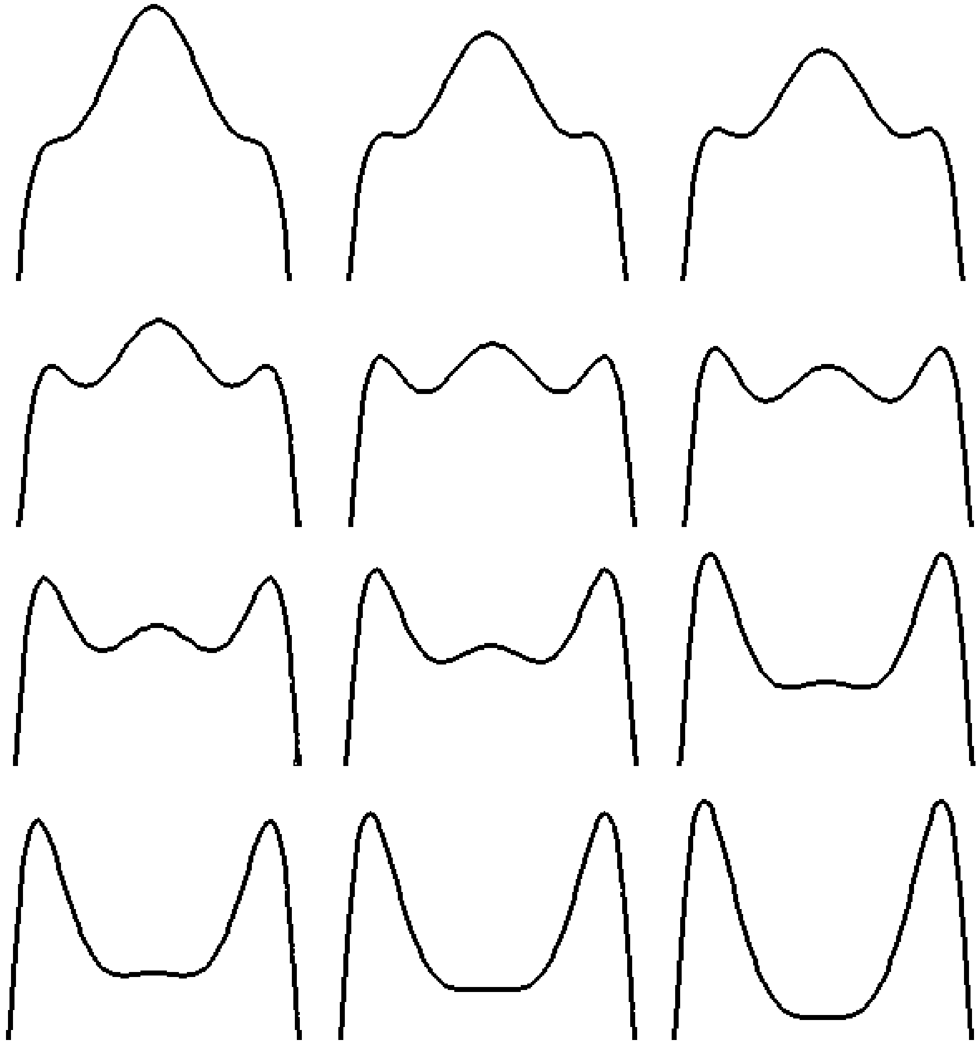
--- $\hat{\hat{f}}_1$

— $\hat{\hat{f}}_2$: shows minimum on the curve $t_{12}(\alpha)$

Potential well $U \propto x^{5.5}$, $\alpha = 1.2$. Trimodal transient state in time evolution of the PDF



Potential well $U \propto x^{5.5}$, $\alpha = 1.2$. Trimodal tra nsient state in a finer scale



Unimodal – bimodal transition in anharmonic potential well

$$U = \frac{ax^2}{2} + \frac{bx^4}{4}, \quad a \geq 0\,, b > 0$$

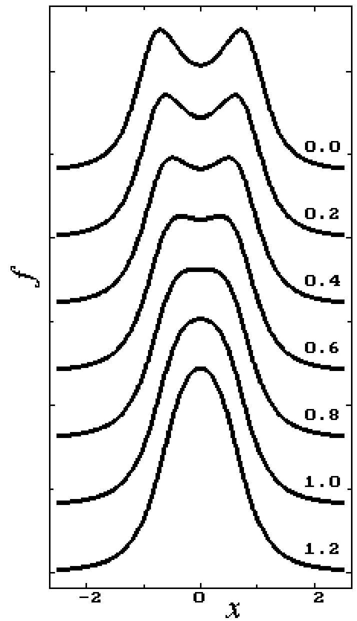
We already know that

- $b = 0$, harmonic $\Rightarrow$ unimodal stationary PDF
- $a = 0$, quartic $\Rightarrow$ bimodal stationary PDF $\Rightarrow$ unimodal-bimodal transition in stationary state while changing parameters ***a*** and/or ***b***

$\frac{d^3 \hat{f}(k)}{dk} - a\frac{d\hat{f}}{dk} = \mathrm{sgn}\, k\, |k|^{\alpha-1}\, \hat{f}(k)$. Transition at $\left.\frac{d^2 f}{dx^2}\right|_{x=0,\, a=a_c} = 0$, or , equiv., $J(a) = \int_0^\infty dk k^2 \hat{f}(k)$ $\quad J(a_c) = 0$

In dimensional variables $b_c = 0.794^{-3} a^3 D^{-2}$, for $\alpha = 1$

a = 0, 0.2, 0.4, 0.6 and 0.8



Confined Lévy flights in bistable potential: Kramers' problem

$$\frac{dx}{dt} = -\frac{dU}{dx} + Y_\alpha(t) \qquad U(x) = -x^2/2 + x^4/4$$

$$T_{Gauss}(D) \propto \exp\left(\frac{1}{4D}\right), \; D << 1$$

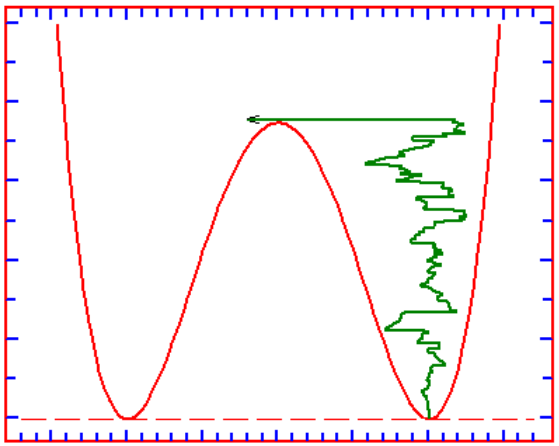


Fig.1. Escape of the trajectory over the barrier, schematic view.

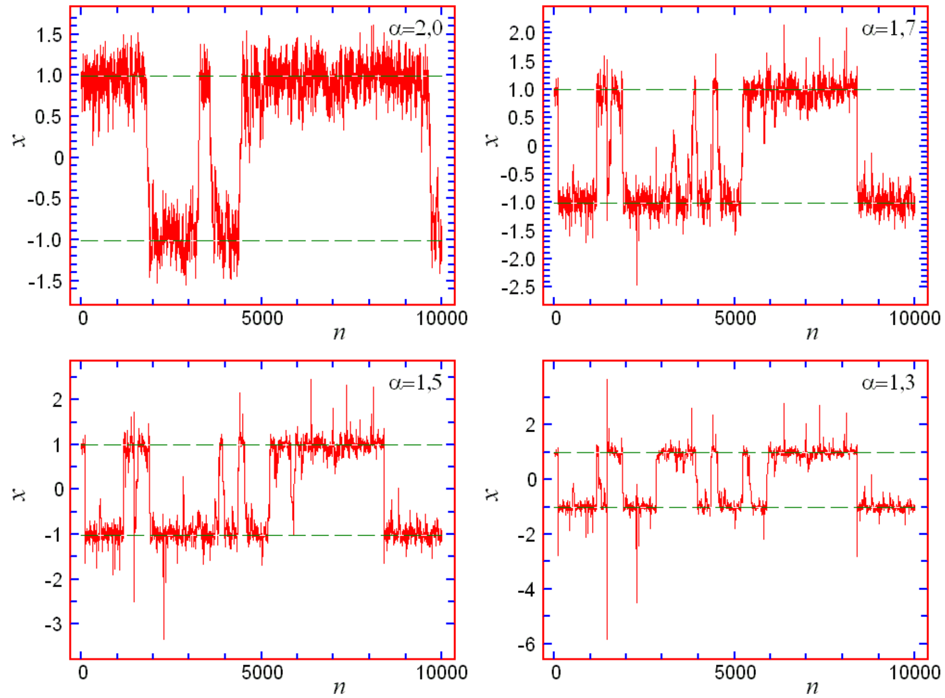


Fig.2. Typical trajectories for different α.

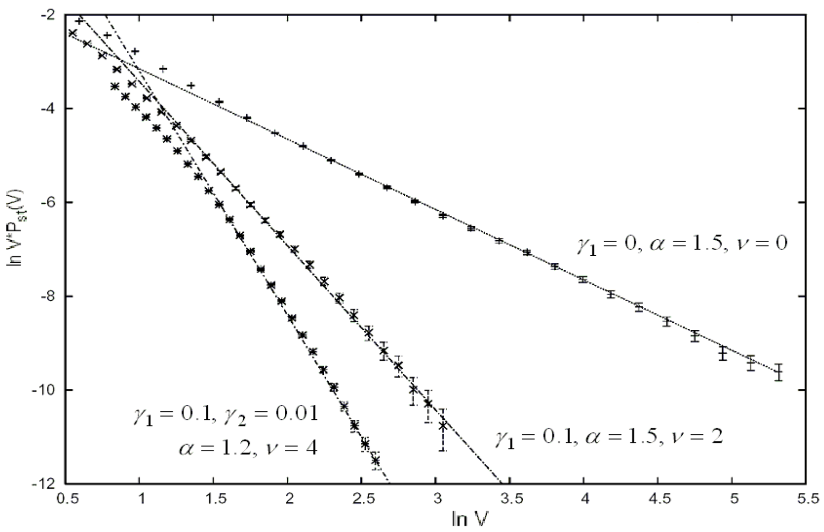
Part II. Damped Lévy flights. Damping by dissipative non-linearity

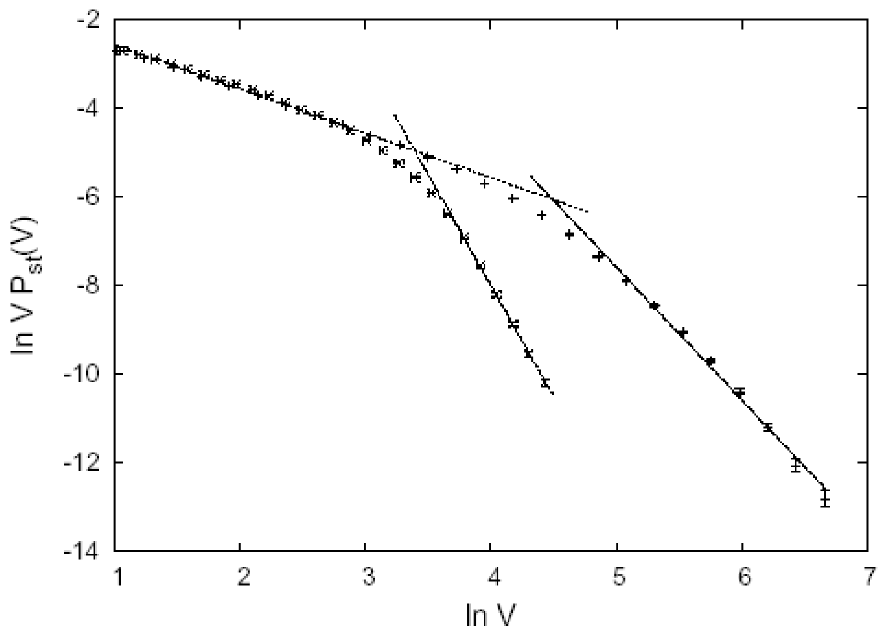
(posed by B. West, V. Seshadri (1982))

$$\textit{Langevin equation} \;\; V(dt) + \gamma(V)V(t)dt = L_{\alpha}(dt)$$

$$\textit{FFPE} \qquad \frac{\partial P(V,t)}{\partial t} = \frac{\partial}{\partial V}\left(V\gamma(V)P\right) + D\frac{\partial^{\alpha} P}{\partial |V|^{\alpha}}$$

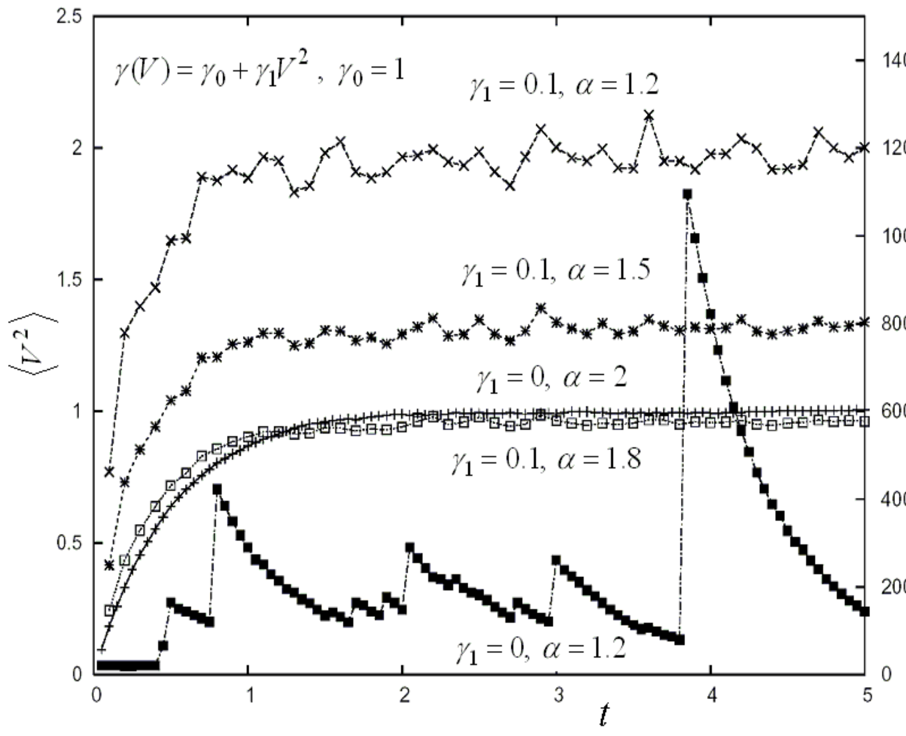
$\gamma(V) = \gamma(-V) = \gamma_0 + \gamma_1 V^2 + \gamma_2 V^4 , \quad \gamma_n > 0$	$\gamma_0 = 1.0$, $\gamma_1 = 0.0001$, $\gamma_2 = 0$ (curve 1) $\gamma_0 = 1.0$, $\gamma_1 = 0$, $\gamma_2 = 0.000001$ (curve 2), $\alpha = 1.0$, slopes = -1, -3, -5



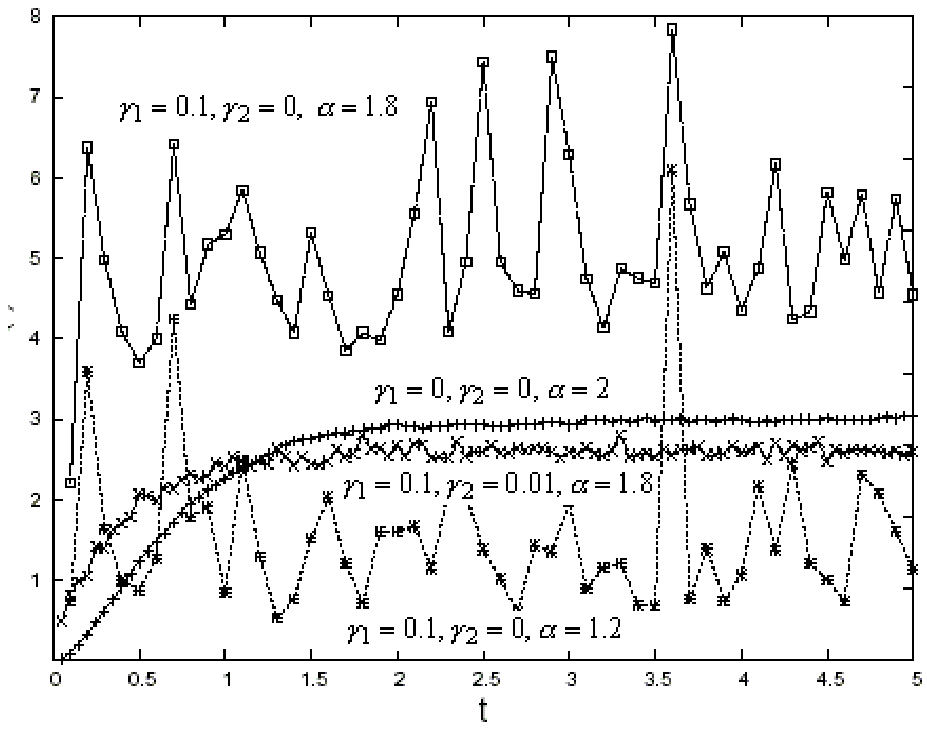


Damped Lévy flights:

2nd Moments



4th Moments



Part III. Power-law truncated Lévy flights

• **"PLT Lévy flights"** : PDFs resembles Lévy stable distribution in the central part

• at greater scales the asymptotics decay in a power-law way, but faster, than the Lévy stable ones, therefore, $\left\langle x^2 \right\rangle < \infty$ $\Rightarrow$ the Central Limit Theorem is applied $\Rightarrow$ • at large times the PDF tends to Gaussian, however, *sometimes very slowly*

Particular case of *distributed order space fractional diffusion equation*:

$$\left(1 - C_\alpha \frac{\partial^{2-\alpha}}{\partial |x|^{2-\alpha}}\right)\frac{\partial f(x,t)}{\partial t} = D\frac{\partial^2 f(x,t)}{\partial x^2} \quad , \qquad 0 < \alpha < 2 \;\Rightarrow\; \left\langle x^2(t) \right\rangle = -\left.\frac{\partial^2 \hat{f}}{\partial k^2}\right|_{k=0} = 2Dt \quad .$$

$$f(x,t) \simeq \frac{\Gamma(5-\alpha)\sin\left(\pi\alpha/2\right)}{\pi}\frac{DC_\alpha t}{x^{5-\alpha}}, \quad x \to \infty$$

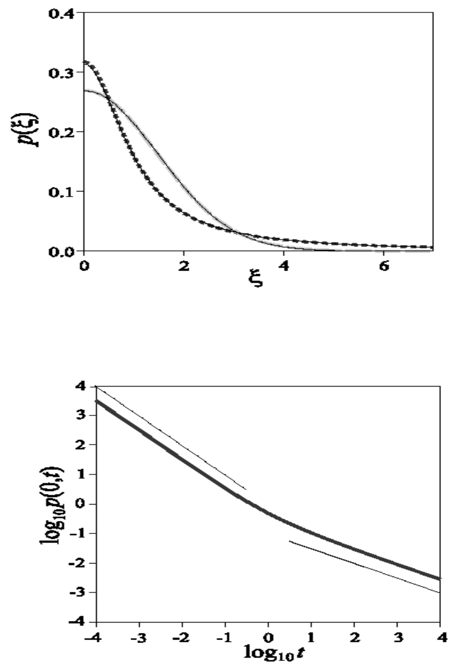
The Lévy distribution is truncated by a power law with a power between 3 and 5. Due to the finiteness of the second moment the PDF *f(x,t)* slowly converges to a Gaussian

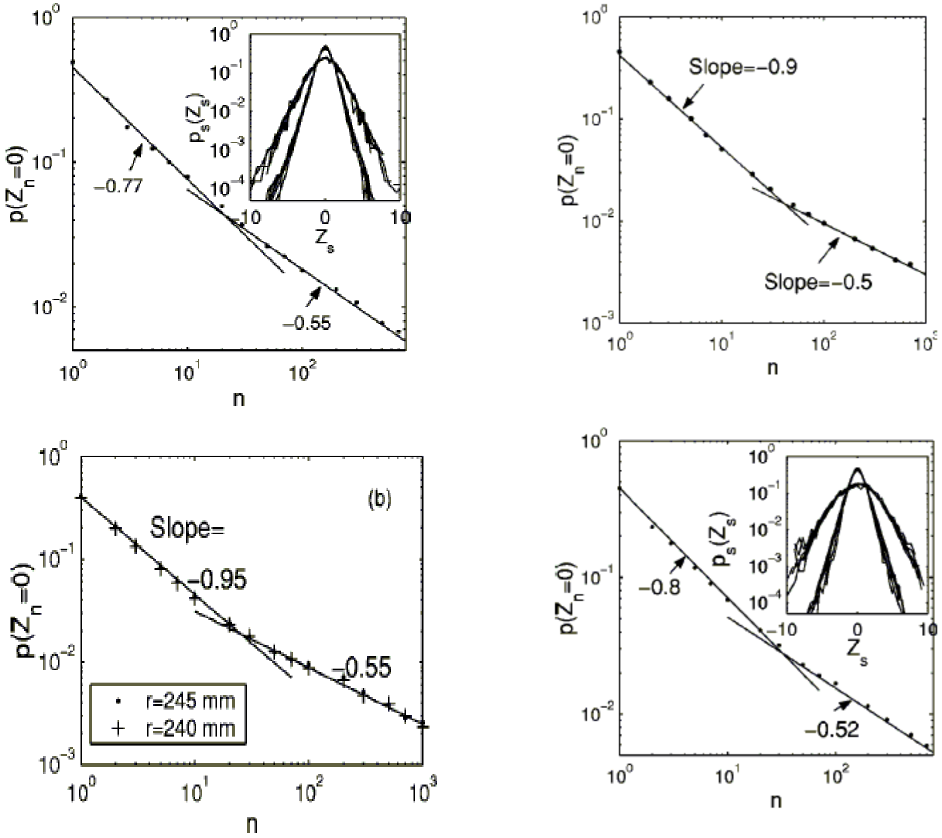
• Probability of return to the origin: $f(0,t) = \int\limits_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-|k|^\alpha t} \propto t^{-1/\alpha} \quad , \quad 0 < \alpha \le 2$.

The slope $-1/\alpha$ in the double logarithmic scales, -1/2 for the Gaussian

Theory

Experiment on the ADITYA tokamak

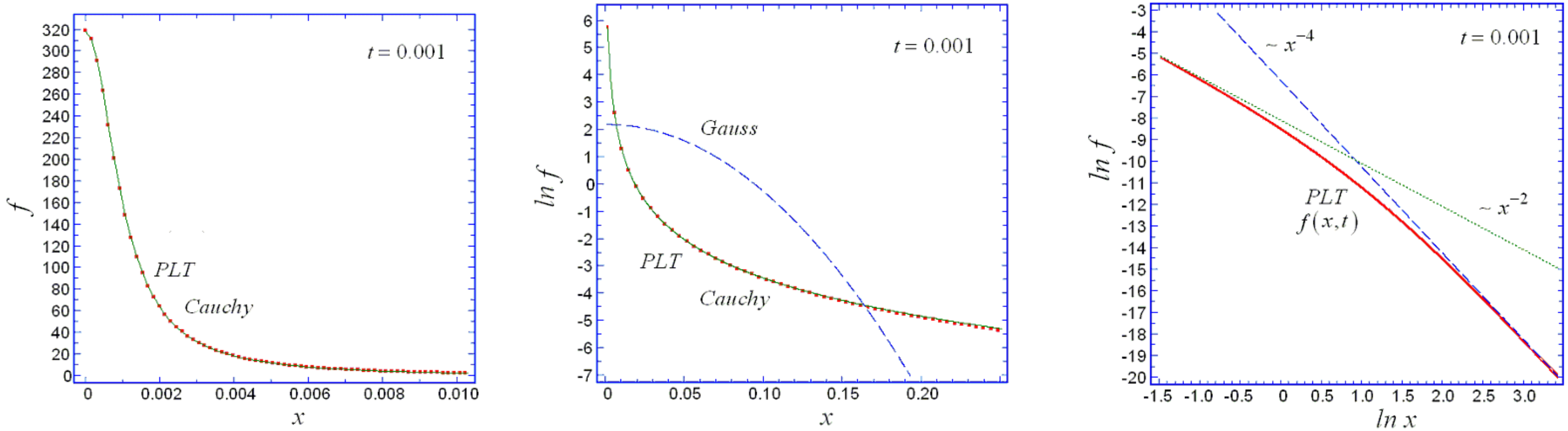




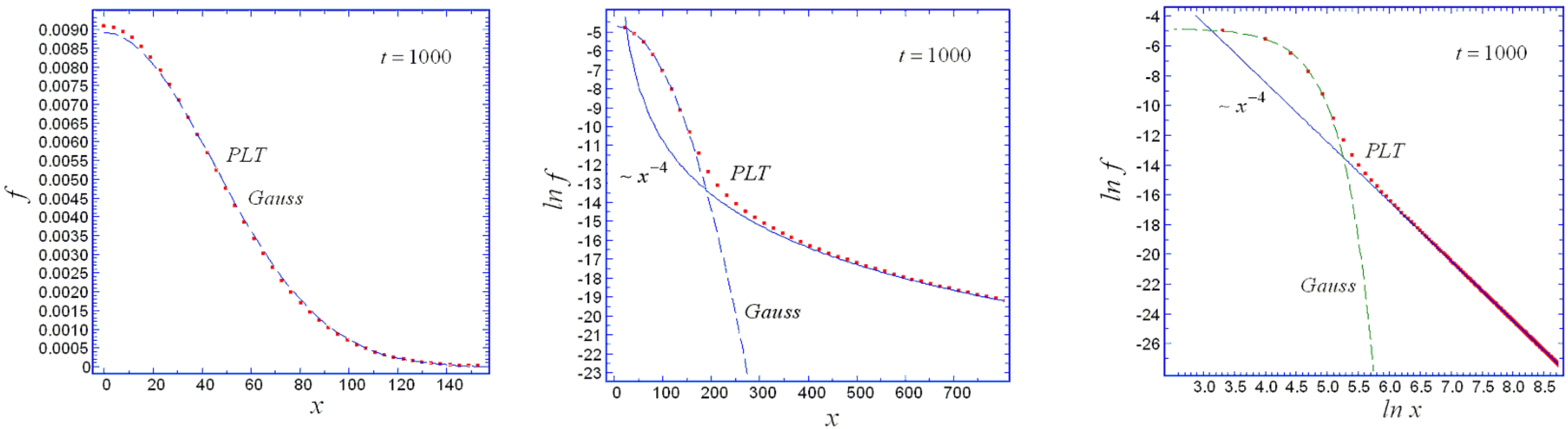
Experiment on the ADITYA: probabilities of return to the origin for different radial positions of the probes **[R.Jha et al.** Phys. Plasmas.2003.Vol.10.No.3.PP.699-704]. Insets: rescaled PDFs.

Evolution of the PDF of the PLT Lévy process

PDF at small time = 0.001

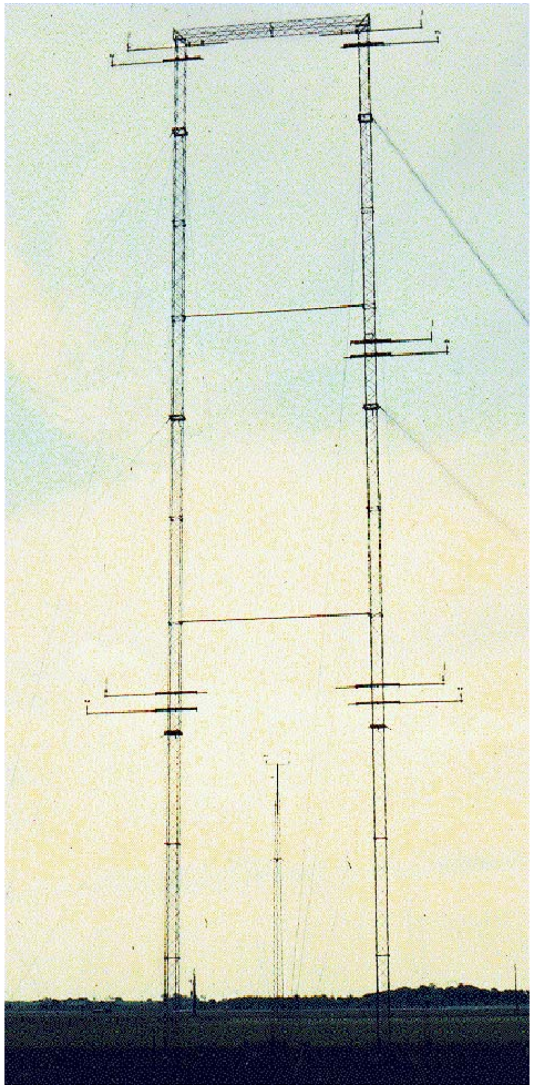


PDF at large time = 1000

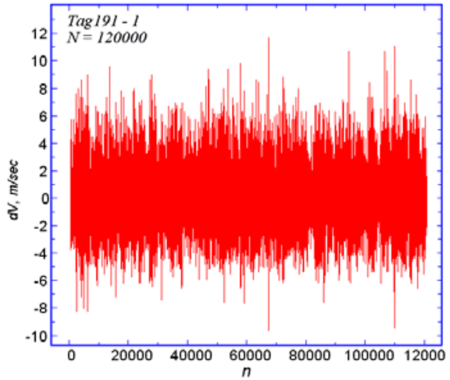


Experimental evidence of PLT Lévy flights in the wind speed measurements (Lammefjord, Denmark)

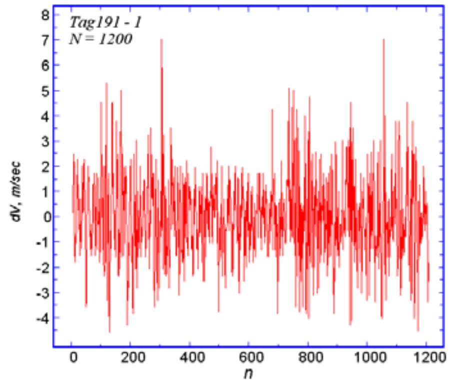
View of measuring masts



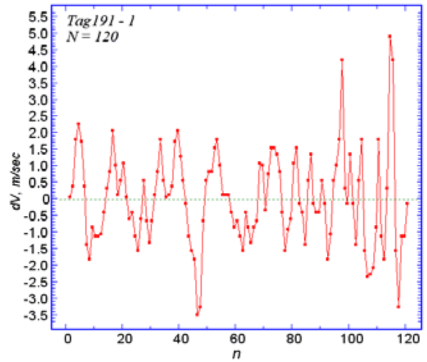
Increments of wind speed



250 min

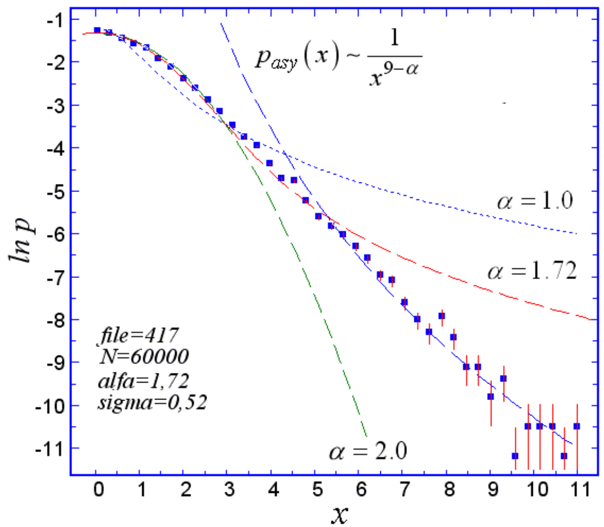


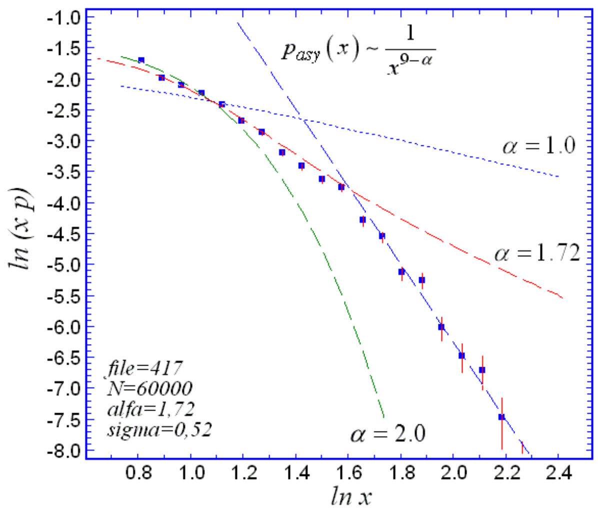
25 min



2.5 min

PDF of the increments:





Short Σ

• Scale-free models provide an efficient description of a broad variety of processes in complex systems

• This phenomenological fact is corroborated by the observation that the power-law properties of Lévy processes strongly persist, mathematically, by the existence of the Generalized Central Limit Theorem due to which Lévy stable laws become fundamental

• All naturally occurring power laws in fractal or dynamic patterns are finite

• Here we present examples of regularisation of Lévy processes in presence of

------ non-dissipative non-linearity : confined Lévy flights,

------ dissipative non-linearity : damped Lévy flights,

------ power law truncation of free Lévy flights

• these models might be helpful when discussing applicability of the Lévy random processes in different physical, chemical, bio..... , financial, … systems