Anomalous diffusion in generalised Ornstein-Uhlenbeck processes

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Abstract

I shall discuss a physically significant extension of a classic problem in the theory of diffusion, namely the Ornstein-Uhlenbeck process [1]. The generalised Ornstein-Uhlenbeck process includes a force which depends upon the position of the particle, as well as upon time. In situations where the damping is weak, so that the particle reaches very large velocities, this results in the momentum distribution satisfying a Fokker-Planck equation of the form

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial p}\left[\left(\gamma p + D(p)\frac{\partial}{\partial p}\right)P\right] \tag{1}$$

where

$$D(p) = D_0(p_0/|p|)^\zeta \tag{2}$$

(where D_0, p_0 are constants and $\zeta \geq 0$). The values $\zeta = 1, 3$ occur for generic forces and smoothly correlated potential forces respectively. This system exhibits anomalous diffusion at short times:

$$\langle x^2(t)\rangle \sim t^{\frac{6+2\zeta}{2+\zeta}} \tag{3}$$

and at long time there is a non-Maxwellian velocity distributions in equilibrium.

Two approaches are used. Some statistics are obtained from a closed-form expression for the propagator of the Fokker-Planck equation for the case where the particle is initially at rest. In the general case we use spectral decomposition of a Fokker-Planck equation, employing a novel type of nonlinear creation and annihilation operator to generate the spectrum, which consists of two staggered ladders.

The anomalous diffusion in our model is not a result of power-law distributions (such as those of the step lengths or waiting times [2]) that are built into the microscopic description. The results are summarised in [5] and discussed fully in [4]. Results on anomalous diffusion in a related model were obtained in [5].

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[3] E. Arvedson, M. Wilkinson, B. Mehlig and K. Nakamura, *Phys. Rev. Lett.*, **96**, 030601, (2006).
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[5] L. Golubovic, S. Feng, and F.-A. Zeng, *Phys. Rev. Lett.*, **67**, 2115, (1991); see also M. N. Rosenbluth, *Phys. Rev. Lett.*, **69**, 1831, (1992).