Solving relaxation and diffusion equation via numerical method

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Some numerical methods for solving differential equations with fractional derivatives, especially Abel-Volterra integral equations are considered in [3], p. 277-290. Here we now give a completely new approach based on the approximation of the tempered convolution with Laguerre series expansions of their elements in the distributional space $\mathcal{S}'_+$, which transforms it in linear system of algebraic equations.

A survey concerning the equations of distributed fractional relaxation and oscillation is treated in [2]. The distributed order fractional diffusion equation and diffusion-like equation with time fractional derivative of distributed order for the kinetic description of anomalous diffusion and relaxation phenomena which describes the sub-diffusion random process subordinated to the Wiener process describing retarding sub-diffusion, is considered in [1]. In [5] are considered the single-order and the double-order fractional relaxation equations as well as the uniformly distributed order fractional relaxation equation. In [11], the multi-dimensional random walk models governed with distributed fractional order differential equations and multi-term fractional order differential equations are constructed. The fundamental solution for the Cauchy problem for a generalized diffusion equation derived from a fractional Fick law describing transport processes with long memory is derived in [4].

All of these equations with fractional derivatives can be solved numerically by the method of the approximation of the tempered convolution with Laguerre polynomials. We shall give here two examples of such kind of equations which can be solved by the method of approximating the tempered convolution with Laguerre polynomials: equation of the relaxation and oscillation (1) and the generalized partial fractional equations with distributed order (2).

The method of approximation of convolution via Laguerre polynomials for $\alpha \in \mathbf{R}$, in the space of tempered distributions with support on $[0,\infty)$, (space $\mathcal{S}'_+$), is developed in [6]. It is applied to solutions of a class of convolution equations, by transforming them to corresponding systems of algebraic equations for the coefficients. The space $\mathcal{S}'_+$ is splitted into the scale of the spaces $\mathcal{S}'_+ = \cup_{s\geq 0} LG'_{0s}$. That gives an approximation for generalized functions, such as delta distributions, its derivatives and multipliers of powers $f'_\alpha s$, the functions $x_+^{\alpha-1}$, $\alpha \in \mathbf{R}\setminus\{-\mathbf{N}_0\}$ and their derivatives of delta's if $\alpha = 0, -1, -2, ...$ The evaluations of such distributions are given in [9].

In fractional calculus we are especially interested in cases $0 < \alpha < 1$ and $1 < \alpha < 2$. Using this method, the coupled system of equations which is a model of viscoelastic rod is solved in [8]. These evaluations in solving, Abel-Volterra integro-differential equations

$$(P(\delta) + \lambda f_\alpha) * g = h(x),$$

P is a polynomial, $P(\delta) = \sum_{i=0}^m a_i \delta^{(i)}$, $\alpha \in \mathbf{R}$, $\lambda \in \mathbf{C}$, δ is the delta distribution, and

$$f_\alpha = \begin{cases} \dfrac{x_+^{\alpha-1}}{\Gamma(\alpha)}, & \alpha > 0, \\ f_{\alpha+N}^{(N)}, & \alpha \leq 0,\ \alpha+N > 0,\ N \in \mathbf{N} \end{cases}$$

$^{(N)}$ is the distributional derivative are given in [8]. For existence-uniqueness result cf. [7].

In this paper, as a model we use the equation of the relaxation and oscillation

$$u(t) = u(0) - \lambda \int_0^t (t-\tau)^{\beta-1} u(\tau) d\tau. \tag{1}$$

We consider pure relaxation $0 < \beta < 1$ and the case of relaxation-oscillation $1 < \beta < 2$, and furthermore the situation of distributed orders integrated with respect to β.

Using this technique we find solutions for the generalized partial fractional equations with distributed order

$$\int_a^b p(x) \frac{\partial^\beta u(t,x)}{\partial t^\beta} d\beta = k^2 \frac{\partial^2 u(t,x)}{\partial x^2}, \quad 0 < a < b < 1, \quad 0 < \beta < 1, \quad (\text{resp. } 1 < \beta < 2). \tag{2}$$

The solution is given by $u(t,x) = \sum_{n=0}^{\infty} \sum_{\bar{n}=0}^{\infty} A_{n,\bar{n}} l_{\bar{n}}(x) l_n(t)$ where $l_n(t) = \exp(-t/2)\ L_n^0(t)$, (resp. $l_{\bar{n}}(x) = \exp(-x/2)\ L_{\bar{n}}^0(x)$), $L_n^0(t)$ (resp. $L_{\bar{n}}^0(x)$) are the Laguerre polynomials. We evaluate coefficients of the Laguerre series $A_{n,\bar{n}}$ from the linear system of equations obtained by evaluation of the convolution equation with respect to the time and space variable separately. Distributional errors for this kind of approximation can be found in [8].

This method (w.r.) to two variables the spatial and the time variable is applied in [10]. For the evaluation of Schrödinger kernel into Laguerre polynomials the expansions for $t_+^{\alpha-1}$, $\alpha \in \mathbf{R}$, from [9], is used to split the space of tempered distributions into the scale of spaces $\mathcal{S}'_+ = \bigcup_{s\geq 0} LG'_{0s}$ (cf. [9]). We determine the vector spaces to which solution belongs, (w.r.) to the time variable t and the space variable x, separately.

References

[1] Chechkin A. V., Gorenflo R., Sokolov I.M., Gonchar V. Yu., Distributed order time fractional diffusion equation. Fractional Calculus and Applied Analysis **6** (2003), 259 - 279.

[2] Gorenflo R., Mainardi F., Fractional Calculus: Integral and differential equations of fractional order. In: A. Carpinteri and F. Mainardi (Editors): Fractals and Fractional Calculus in Continuum Mechanics, pp. 223 - 176, Springer Veralag, Wien and New York, (1997).

[3] Gorenflo R., Mainardi F., Fractional relaxation of distributed order. In M. Novak (Editor): "Complexus Mundi: Emergent Patterns in Nature", World Scientific, Singapore, 2006, pp. 33-42. Proceedings of the Conference "FRACTAL 2006", Vienna 12-15 February (2006).

[4] Mainardi F., Gorenflo R., Sub-diffusion equations of fractional order and their fundamental solutions, Mathematical methods in engineering, International symposium, Ankara, Turkey, April 27-29, (2006).

[5] Mainardi F., Fractional relaxation of distributed order, 9^{th} international conference complexity and fractals in nature, Fractal 2006, Vienna, Austria, 12-15 February, (2006). Preprint.

[6] Pilipović S., Stojanović M., Solving Convolution Equations in S'_+ by Numerical Method, Publ. Math. Debrecen, **40**/1-2, 17-33, (1992).

[7] Pilipović S., Stojanović M., Volterra-type equations in the space $\mathcal{S}'_+$, (Russian) Mat. Zametki **75**, no. 1, 135-141, (2004); translation in Math. Notes **75**, no. 1-2, 124-130, (2004).

[8] Pilipović S., Stojanović M., Fractional differential equations through Laguerre expansions in abstract spaces. Error estimates, Integral Transf. Spec. Funct. Preprint (2006).

[9] Stojanović M., Splitting $\mathcal{S}'_+$ space into union of Banach spaces through the Laguerre expansions, Preprint (2006).

[10] Stojanović M., Nonlinear Schrödinger equation with singular potential and initial data in vector spaces with Laguerre expansion, Preprint (2006).

[11] Umarov S., Steinberg S., Random walk models associated with distributed fractional order differential equations, IMS Lecture Notes-Monograph series, imsart-lnms ver. (2005)10/19, March 23, (2006).