International Seminar on Anomalous Transport:
Experimental Results and Theoretical Challenges
Phisikzentrum Bad Honnef (Germany), 12-16 July 2006

Sub-diffusion equations of fractional order and their fundamental solutions

Francesco MAINARDI
Department of Physics, University of Bologna, and INFN,
Via Irnerio 46, I-40126 Bologna, Italy.
E-Mail: `mainardi@bo.infn.it` URL: `http://www.fracalmo.org`

Abstract

The time fractional diffusion equation is obtained from the standard diffusion equation by replacing the first-order time derivative with a fractional derivative of order $\beta \in (0,1)$. From a physical view-point this generalized diffusion equation is derived from a fractional Fick law which describes transport processes with long memory. The fundamental solution for the Cauchy problem is interpreted as a probability density of a self-similar non-Markovian stochastic process related to a phenomenon of sub-diffusion (the variance grows in time sub-linearly).

A further generalization is obtained by considering a continuous or discrete distribution of fractional time derivatives of order less than one. Then the fundamental solution is still a probability density of a non-Markovian process that, however, is no longer self-similar but exhibits a corresponding distribution of time-scales. For particular choices of order distributions (discrete ones, continuous ones) explicit model solutions are found, in detail analyzed with respect to asymptotics for time tending to zero and to infinity; numerical-graphical studies are carried out as well.

This lecture is based on Author's works carried out with the collaboration of his associates Dr. Antonio Mura and Dr. Gianni Pagnini, and of his colleague Rudolf Gorenflo, Emeritus Professor of Mathematics at the Free University of Berlin. Below we provide some references to related papers of our team.

References

[1] A. V. Chechkin, R. Gorenflo, I. M. Sokolov, V. Yu. Gonchar, Distributed order time fractional diffusion equation, *Fractional Calculus and Appl. Analysis* **6** (2003) 259-279.

[2] R. Gorenflo, Yu. Luchko and F. Mainardi, Wright functions as scale-invariant solutions of the diffusion-wave equation, *J. Comput. and Appl. Mathematics* **118** (2000) 175-191.

[3] R. Gorenflo and F. Mainardi, Fractional calculus: integral and differential equations of fractional order, in: A. Carpinteri and F. Mainardi (Editors), *Fractals and Fractional Calculus in Continuum Mechanics*, Springer Verlag, Wien (1997), pp. 223–276. [Reprinted in `http://www.fracalmo.org`]

[4] F. Mainardi, Fractional relaxation-oscillation and fractional diffusion-wave phenomena, *Chaos, Solitons and Fractals* **7** (1996) 1461–1477

[5] F. Mainardi, Fractional calculus: some basic problems in continuum and statistical mechanics, in: A. Carpinteri and F. Mainardi (Editors), *Fractals and Fractional Calculus in Continuum Mechanics*, Springer Verlag, Wien and New-York (1997), pp. 291–248. [Reprinted in `http://www.fracalmo.org`]

[6] F. Mainardi, Yu. Luchko and G. Pagnini, The fundamental solution of the space-time fractional diffusion equation, *Fractional Calculus and Appl. Analysis* **4** (2001) 153-192. [Reprinted in `http://www.fracalmo.org`]

[7] F. Mainardi and G. Pagnini, The Wright functions as solutions of the time-fractional diffusion equations, *Appl. Math. and Comp.* **141** (2003) 51-62.

[8] F. Mainardi and G. Pagnini, The role of the Fox-Wright functions in fractional subdiffusion of distributed order, *J. Comput. and Appl. Mathematics* (2006), in press.