On the Distinguished Role of the Mittag-Leffler Waiting Time Density

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Abstract: The entire function $E_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+k\beta)}$ and some of its generalizations in recent two decades gradually have come to be recognized as useful in diverse applications. Some such applications have been discovered first in the Laplace transform domain (Gnedenko and Kovalenko 1968, Balakrishnan 1985), the authors not seeing that in the time domain variants of the Mittag-Leffler function enter the game, namely, for $0 < \beta \le 1, \quad t > 0$, the functions $\Psi_\beta(t) = E_\beta(-t^\beta)$ and $\psi_\beta(t) = -\Psi_\beta{}'(t)$ which both are completely monotone and appear in processes of *fractional relaxation*. These functions also play decisive roles as (residual) waiting time probability distribution or density in the theory of *continuous time random walks* (Hilfer and Anton 1995) and appear as the limiting law in *infinite thinning* of a renewal process for an initial density with power law asymptotics (Gnedenko and Kovalenko 1968). We outline these models and show furthermore that the Mittag-Leffler waiting time law is the precise description of the properly rescaled long time behaviour of a renewal process whose waiting time density is long-tailed due to a power law decay. In this sense the Mittag-Leffler renewal process, generalizing the classical Poisson process, is an important limiting process. In the analysis of such asymptotic behaviour the Mittag-Leffler density $\psi_\beta(t)$ exhibits stability against rescaling combined with a deceleration and a characteristic kind of self-similarity. Remarkably the same transformation formula arises in the analysis of infinite thinning and of long time behaviour of a *power law renewal process*. It is instructive to consider variants of passing to the diffusion limit (in space only or in time only or in both simultaneously, the time-fractional drift process) and to look to *distributed fractional order* processes and their relation to *multiply scaled* continuous time random walks.
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References

[1] V. Balakrishnan: Anomalous diffusion in one dimension. Physica 132A (1985), 569-580.
[2] A. V. Chechkin, R. Gorenflo, I. M. Sokolov, V. Yu. Gonchar: Distributed order time-fractional diffusion equation. *Fractional Calculus and Applied Analysis* **6** (2003), 259 - 279.
[3] B. V. Gnedenko and I. N. Kovalenko: Introduction to Queueing Theory. Israel Program for Scientific Translations, Jerusalem 1968.
[4] Rudolf Gorenflo and Francesco Mainardi: Simply and multiply scaled diffusion limits for continuous time random walks. In: S. Benkadda, X. Leoncini and G. M. Zaslavsky (editors): *Chaotic Transport and Complexity in Fluids and Plasmas. Journal of Physics: Conference Series (JPCS)* **7** (2005), 1-16. Institute of Physics Publishing.
[5] R. Gorenflo and Entsar A. Abdel-Rehim: From power laws to fractional diffusion: the direct way. Vietnam Journal of Mathematics **32 SI** (2004), 65 - 75.
[6] R. Hilfer: On fractional diffusion and continuous time random walks. *Physica A* **329** (2003), 35 - 39.
[7] R. Hilfer and L. Anton: Fractional master equation and fractal time random walk. Physical Review E **51** (1995), R848-R851.
[8] F. Mainardi, R. Gorenflo, E. Scalas: A fractional generalization of the Poisson processes. Vietnam Journal of Mathematics **32 SI** (2004), 53 - 64.